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Generalizing The Diagonal Approximated Signature To Systems With Multiple Component Types

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Abstract

The ever-increasing scale and complexity of modern systems presents engineers with the inevitable challenge of developing more efficient yet comprehensive computational tools that enable sound analysis and thus ensure reliable system operation. In this paper, the Continuous-State Survival Function (CSSF) of coherent systems is investigated, and the Diagonally Approximated Signature (DAS) is generalised to systems with multiple component types and utilized as a corresponding surrogate model. The proposed concept is based on a combinatorial decomposition and aggregation strategy that is adapted from the concept of survival signature. This provides the advantageous property of separating topological and probabilistic information and component probability structure. Potentially high-dimensional coherent structure functions form the basis of the analysis. The proposed approach enables the direct computation of the CSSF using explicit formulas and a stored DAS, avoiding the costly online Monte Carlo simulation (MCS) in repeated model evaluations for varying component probability structures. The case studies underline the theoretical findings and, in particular, the high computational efficiency of the concept of the condensed DAS.

Keywords: reliability analysis, continuous-state systems, continuous-state survival function, coherent structure function, surrogate modeling

1. Introduction

Engineering systems, encompassing infrastructure networks, industrial plants, and complex machinery, are pivotal to the advancement and development of modern societies. These systems, deeply integrated into societal frameworks, significantly influence both the economy and daily life. However, they are subject to degradation due to environmental and operational factors, leading to diminished performance or, in extreme cases, complete failure. Therefore, ensuring the continuous functionality and reliability of these systems is critical for both economic and safety reasons. This necessitates informed decision-making in their design and maintenance, especially as these systems grow in size and complexity in an increasingly dynamic and unpredictable world. To achieve this, accurately assessing the system's reliability over time is crucial. The structure-function is a typical approach to modelling the interaction of components and their relation to system functionality (Gámiz and Miranda, 2010; Coolen and Coolen-Maturi, 2016). Thereby, multi- and continuous-state considerations become increasingly important to achieve a better understanding of the system behavior and achieve more safety and higher functionality with fewer resources (Lisnianski and Frenkel, 2012; Cellier and Greifeneder, 2013). Recent approaches include advancements in the modelling of continuous-state degradation processes, compare (Kharoufeh and Cox, 2005; Giorgio et al., 2011; Rui et al., 2020).

An efficient method for modelling the reliability of systems with various component types is the concept of survival signature, as developed by Coolen and Coolen-Maturi (Coolen and Coolen-Maturi, 2013, 2016). This concept stands out from traditional methods by distinctly separating the topological aspects of the system from the probabilistic characteristics of its components. The separation simplifies the computational process, which is often required in design and maintenance, as highlighted by Patelli et al. (Patelli et al., 2017). This efficiency is

particularly beneficial in reducing computational demands during repeated model evaluations, bypassing the need for extensive and redundant analysis of the topological system model.

However, the original form of the survival signature has a limitation: it only considers binary states of components and systems. Considering the continuous performance states of components and systems is essential for thorough reliability analyses of real-world systems. To address this, several researchers developed approaches for considering discrete or continuous-state multi-state systems, cf. (Eryilmaz and Tuncel, 2016; Liu et al., 2018; Qin and Coolen, 2022). Further, Winnewisser et al. introduced a novel surrogate modelling approach (Winnewisser et al., 2023), referred to as the concept of diagonally approximated signature (DAS), based on an unconventional shift of the perspective and modelling framework. This approach builds on the survival signature concept but further enables the consideration of high-dimensional coherent structure functions that describe the interplay between continuously deteriorating components and the continuous-state performance of the system. This method allows for the direct computation of a continuous-state survival function utilising an explicit formula and a stored DAS. This approach not only bypasses the need for online Monte Carlo simulations but also overcomes the binary state limitation, offering a more comprehensive analysis of system reliability.

The DAS concept proposed in (Winnewisser et al., 2023 was limited to considering components of a single type. However, this constraint is inadequate for real-world applications, where systems often comprise multiple types of components (Feng et al., 2016). In order to address this gap, this paper introduces novel developments, extending the concept of DAS to systems with multiple component types. An aggregation method within the DAS framework condenses the fundamental DAS and facilitates tremendously increased efficiency. The proposed developments not only enhance the accuracy of reliability assessments but also contribute to more effective maintenance and design strategies, crucial for the sustainable operation of modern engineering systems.

The paper is structured as follows: Section 2 briefly summarises the theoretical fundamentals. The proposed methodology, including the reformulated and extended formulas for computing the survival function when pursuing a continuous-state perspective, are presented in Section 3. Further, in Section 4, the efficiency and accuracy of three different approaches is investigated in two case studies. Section 5 concludes the findings and provides possible future directions to further enhance the concept of DAS.

1. Theoretical Fundamentals

1.1. Structure Function

The structure function of a system is a fundamental concept for modelling the system topology in reliability analysis. As basis for the proposed methodology, let there be a structure function $\phi(\boldsymbol{x})$ that describes the system topology, i.e., it maps the component states to the system state. Assume that the considered structure function is time-invariant. It can be defined in a binary-state context as $\phi(\boldsymbol{x})\colon \{0,1\}^n \rightarrow \{0,1\}$. Thereby, $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ is the according state vector representing the potentially time-dependent and random state of n components. This consideration is the basis for the considerations in (Coolen and Coolen-Maturi, 2013). In the discrete multi-state context as $\phi(\boldsymbol{x})\colon \{0,1, \dots, m^c\}^n \rightarrow \{0,1, \dots, m^s\}$, compare (Eryilmaz and Tuncel, 2016), or from a continuous-state perspective as $\phi(\boldsymbol{x})\colon [0,1]^n \rightarrow [0,1]$. see (Gámiz and Miranda 2010).

Binary-state, discrete multi-state, as well as continuous multi-state structure functions can be coherent. In accordance with the definition in (Hudson and Kapur 1983), a system is coherent if $\phi(\boldsymbol{x})$ is surjective, ϕ is monotone and non-decreasing, i.e., $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$ if $\boldsymbol{x} \leq \boldsymbol{y}$, and there are no inessential components, i.e., each component influences the system performance at some point.

1.2. Continuous-State Survival Function

In this work, the probability that the random state variable E_s of some entity under consideration is greater or equal to the considered state s at given time t is investigated and hereafter referred to as the Continuous-State Survival Function (CSSF) of the according entity. It is denoted by

$$R_{E_s}(s,t) = R_{E_s}(s|t) = P(E_s \geq s|t), \tag{1}$$

where $E_s$is the random system or component performance state variable in the interval [0,1]. Note that $R_{E_s}(s,t)$ can also be characterized in terms of the random failure time variable E_t given a considered performance state s. However, it is exactly this shift from the consideration of random failure times with respect to some state to the consideration of random performance states with respect to some point in time that allows for the proposed

developments. For more detailed information on the interpretation and establishment of the CSSF, see (Winnewisser et al. 2023).

1.3. Limit State Analyses for Evaluating Discrete Multi-State System Reliability

Consider a system with a coherent, time-invariant, and continuous-state structure function $\phi(\boldsymbol{x})$ as discussed in section 2.1. Assume that the performance state x_i of the i-th component with $i = 1,2,\dots,n$ is random at given time t and follows the probability density function $f(x_i \mid t)$. Accordingly, $f(\boldsymbol{x} \mid t)$ denotes the joint probability density function characterizing the random component performance state vector $\boldsymbol{x} = [x_1, x_2, \dots, x_n]$. Note that $\int_{\mathrm{X}_{i,s}} f(x_i \mid t)d\mathrm{x} = R_{x_i}(s \mid t) = 1 - F(x_i \mid t)d\mathrm{x} = 1 - \int_{1-\mathrm{X}_{i,s}} f(x_i \mid t)d\mathrm{x}$, where $\mathrm{X}_{i,s}$ refers to the domain in the interval $\mathrm{X}_i = [0,1]$ with values that are greater than state s. Correspondingly, the overall domain for the component state vector $\boldsymbol{x}$ is $\Omega = \mathrm{X}_1 \times \mathrm{X}_2 \times \dots \times \mathrm{X}_n = [0,1]^n$.

The aim of the discrete multi-state reliability analysis is the computation of $R_{\phi_s}(s,t)$ that is the CSSF of the considered system with respect to the underlying structure function $\phi(\boldsymbol{x})$ given some component probability structure. Hereafter, denote $R_{\phi_s}(s,t)$ as $R(s,t)$ and recall $\Omega_s = \{\boldsymbol{x} | \phi(\boldsymbol{x}) \geq s\}$. Then, the CSSF can be evaluated by integrating the joint probability density function $f(\mathbf{x} \mid t)$ over the domain Ω_s that meets the threshold for the system performance:

$$R(s,t) = \int_{\Omega_s} f(\mathbf{x} \mid t)d\mathbf{x} = \int_{\Omega} I\left(\phi(\mathbf{x}) \geq s\right) f(\mathbf{x} \mid t)d\mathbf{x}. \tag{2}$$

The domain considered for integration can be extended to the overall domain Ω by weighting this expression with the indicator function. This corresponds to a limit state analysis that also can be expressed as Monte Carlo Simulation (MCS) as

$$R(s,t) = \frac{1}{N_{MCS}} \sum_{j=1}^{N_{MCS}} I\left(\phi\left(\mathbf{x}_j(t)\right) \geq s\right), \tag{3}$$

where $\mathbf{x}_j(t)$ is the j-th state vector sample of overall N_{MCS} Monte Carlo samples from the joint probability distribution $f(\mathbf{x} \mid t)$ for given point in time t.

1.4. Classifications of the Structure Function from a Diagonal Perspective

Considering a coherent structure function that maps the continuous-state performance state of n components to the continuous-state performance state of the system, two properties with multiple categorisations are presented:

At first, the diagonal state sign is assigned as diagonally state positive if it holds that $\phi(\boldsymbol{x}_d) > s \; \forall s \in [0,1]$ with $\boldsymbol{x}_d = (x_1, x_2, \dots, x_n)$ and $x_i = s$. Analogously, the terms diagonally state neutral and diagonally state negative correspond to the conditions $\phi(\boldsymbol{x}_d) = s$ and $\phi(\boldsymbol{x}_d) < s$, respectively. Fig. 1 illustrates these various assignments

of the diagonal state sign for four arbitrary systems. A coherent continuous-state structure function can also be partly diagonally state positive, partly neutral and partly negative, cf. Fig. 1.

Secondly, the coherent structure function can be classified in terms of its symmetry along $\boldsymbol{x}_d$ for $x_i = s$ and $\forall s \in [0,1]$. Fig. 2. shows contour plots of a symmetric coherent structure function (a) and an unsymmetric coherent structure function (b).

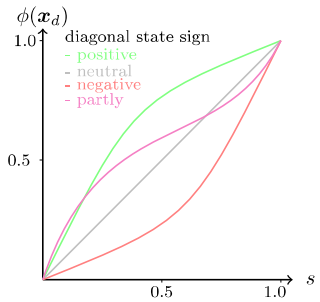

Fig. 1. Various possible assignments of the diagonal state sign.

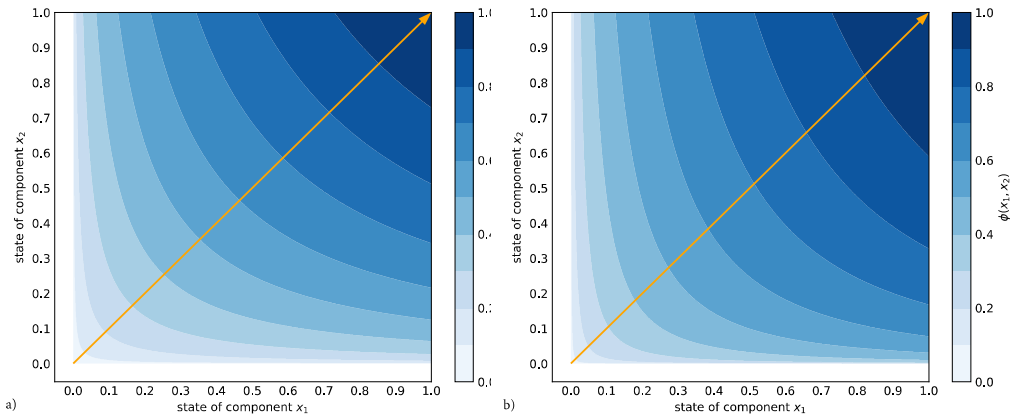

Fig. 2. Contour plots of a system that is (a) diagonally symmetric and (b) diagonally unsymmetric.

2. Proposed Methodology: The Concept of Diagonally Approximated Signature

The concept of the Diagonally Approximated Signature (DAS) was first presented in (Winnewisser et al. 2023) as a surrogate modelling approach for evaluating the CSSF of potentially continuous-state systems more efficiently when it comes to repeated model evaluations. It was inspired by the concept of survival signature introduced in (Coolen and Coolen-Maturi, 2012) and its extensions to discrete multi-state considerations as in (Eryilmaz and Tuncel, 2016) and (Qin and Coolen, 2022) and the continuous multi-state perspective, as in (Liu et al., 2018). All these concepts share the combinatorial decomposition and aggregation of the component state space that enables reduced computational cost. Thereby, the concept of DAS preserves the advantages separation property while being applicable to systems that are described as coherent, discrete or continuous multi-state structure function, mapping component to system performance state. Similar to the other concepts of (survival) signature, the DAS values are evaluated in a pre-processing step and can be reused when investigating the system reliability for various component probability structures. The result of the concept of DAS is the true CSSF or at least an underestimation but never an overestimation with respect to the considered component probability structure. The quality of this result in terms of difference between the true CSSF and via DAS evaluated CSSF is hereafter referred to as error margin and depends on the characteristics of $\phi(\mathbf{x})$ from a diagonal perspective.

2.1. Fundamental Formula for the Consideration of a Single Component Type

The fundamental formula of the concept of DAS is derived from a combinatorial decomposition of the component state space based on the number of components that function in state s or above. This decomposition can be outlined as

$$\Omega_s^{\phi} = \{\boldsymbol{x} | \phi(\boldsymbol{x}) \geq s\} = \bigcup_{l_s}^{n} \Omega_{s,l_s}^{\phi} = \Omega_{s,0}^{\phi} \cup \bigcup_{l_s=1}^{n-1} \Omega_{s,l_s}^{\phi} \cup \Omega_{s,n}^{\phi} = \bigcup_{l_s=0}^{n} \bigcup_{p}^{\binom{n}{l_s}} \Omega_{s,l_s,p}^{\phi}. \tag{4}$$

Thereby, $\Omega_{s,l_s,p}^{\phi}$ refers to all possible combinations of the component state vector $\boldsymbol{x}$ or the hypervolume over $\boldsymbol{x}$ in the subspace $\Omega_{s,l_s,p}$ that fulfills the criteria $\phi(\boldsymbol{x}) \geq s$. This combinatorial decomposition also holds true for the overall domain

$$\Omega = \{\boldsymbol{x}\} = \bigcup_{l_s=0}^{n} \bigcup_{p}^{\binom{n}{l_s}} \Omega_{s,l_s,p}, where\ \boldsymbol{x} \in [0,1]^n, \tag{5}$$

for any considered state s. The binomial coefficient $\binom{n}{l_s}$ is further denoted as $\mathrm{P}(l_s)$ and varies in terms l_s the number of components working in state s or above. $\mathrm{P}(l_s)$ is the number of all possible permutations of $\boldsymbol{x}$ with $x_i \in \{w_s, \widetilde{w}_s\}$, where w_s means that the component is working in state s or above and $\widetilde{w}_s$ means that the component state is smaller than state s. All possible assignments of $\boldsymbol{x}$ corresponding to such a permutation define a subspace $\Omega_{s,l_s,p}$ with respect to the considered state s.

Assume that the structure function of the considered system is coherent and time-invariant and the comprised components have the same type. This means that the random states or failure times are assumed to be iid. Formally this can be described as $R_x(s \mid t) = R_{x_i}(s \mid t)$ for $i = 1,2,\dots,n$. Further, let the structure function be diagonally state neutral over the entire diagonal $\boldsymbol{x}_d$. If the structure function is also diagonally state extreme or at least constant, the concept of DAS gives the true solution of the CSSF. It holds true that

$$R(s,t) = \sum_{l_s=0}^{n} \sum_{p=1}^{\mathrm{P}(l_s)} [R_x(\Phi(s,l_s,p) \mid t) - R_x(s \mid t)]^{n-l_s} [R_x(s \mid t)]^{l_s}, \tag{6}$$

where $R(s,t) = R_{\phi_s}(s,t)$ is the CSSF representing the continuous-state reliability of the considered system. For a more detailed derivation of the combinatorial decomposition and the fundamental formula, see (Winnewisser et al.,2023).

Note that from the continuous-state perspective all conventionally in binary-state system reliability analysis considered structure functions are diagonally state neutral and diagonally state extreme. This corresponds to a simplification of the underlying problem that mitigates a comprehensive analysis in terms of a minimised CSSF error margin. In contrast, the concept of DAS allows for a tradeoff between computational cost and a reduced CSSF error margin when it comes to more complex structure functions that describe the considered problems from a continuous-state perspective.

The DAS values $\Phi(s,l_s,p)$ are evaluated in a pre-processing step by performing $\sum_{s}^{n_{states}} \sum_{l_s}^{n_{states}} P(l_s)$ one-dimensional optimizations along the diagonals defined by the subspaces $\Omega_{s,l_s,p}$. The aim of this optimisation is to find the minimum value $v_{s,l_s,p}$ that still fulfills the criterion $\phi(\boldsymbol{x}) \geq s$ along the diagonal defined by the permutation of $\boldsymbol{x}$ with $x_i \in \{w_s, \widetilde{w}_s\}$, where $w_s = s$ and $\widetilde{w}_s = v_{s,l_s,p} \in [0,s]$. The optimization procedure should minimize the error between the true value $v_{s,l_s,p}^{\phi}$ for which $\phi(\boldsymbol{x}) = s$ and the result of optimization procedure $v_{s,l_s,p}$. A basic algorithm that ensures the evaluated CSSF to underestimate the true CSSF, i.e., to be on the safe side, was presented in (Winnewisser et al., 2023).

2.2. Advantageous Representation and According Formula

The fundamental formula can be reformulated in an advantageous manner to allow for vectorisation of the computation of the CSSF. Therefore, let $\boldsymbol{\Psi} = [\boldsymbol{\Psi}^c, \boldsymbol{\Psi}^v, \boldsymbol{\Psi}^{l_s}]$ be a three-dimensional matrix, where $\boldsymbol{\Psi}^c$ is a two-dimensional matrix that comprises $\sum_{l_s}^{n} P(l_s)$ values of $\Phi(s,l_s,p)$ as rows and for each considered state s as columns. Similarly, $\boldsymbol{\Psi}^c$ counts the occurrences of the values $\Phi(s,l_s,p)$ in $\sum_{l_s}^{n} P(l_s)$ rows and columns for considered state s and $\boldsymbol{\Psi}^{l_s}$ holds the corresponding number of components working in state s or above. In accordance with this representation, the formula for an underapproximation of the CSSF can be given as

$$R(s,t) \geq \sum_{j=1}^{\sum_{l_s}^{n} P(l_s)} \Psi^c(j,s) \, [R_x(\Psi^v(j,s) \mid t) - R_x(s \mid t)]^{n-\Psi^{l_s}(j,s)} [R_x(s \mid t)]^{\Psi^{l_s}(j,s)}. \tag{7}$$

Note that Equation (6) was reformulated to an inequality here. Consequently, the statement holds true for systems with a structure function that is diagonally state positive or at least neutral and an arbitrary diagonal state order, cf. (Winnewisser et al., 2023). Thereby, the CSSF evaluated by means of the concept of DAS underestimates the true CSSF in the worst case.

In addition, the newly introduced representation $\boldsymbol{\Psi}$ was integrated. $\Psi^c(j,s)$ is the only additional parameter compared to the previous formulation and has no advantage yet. For this basic form, $\Psi^c(j,s) = 1$ for every j and s. However, introducing this occurrence counter $\Psi^c(j,s)$ enables to reduce the sum over $\sum_{l_s}^{n} P(l_s)$ tremendously depending the according aggregation strategy.

2.3. Extended Formula for the Consideration of Multiple Component Types

Consider a system with coherent and time-invariant structure function $\phi(\boldsymbol{x})$ and components of multiple types. This means that among the n components n_k components share the same type $k = 1,2,\dots,K$ and that their random performance states at given time t are assumed to be iid. Formally, the CSSFs characterizing the random

performance states for components of the same time can be summarized as $R_{x_k}(s \mid t) = R_{x_i}(s \mid t)$ if the i-th component is of type k. The representation of the DAS should be adapted accordingly. Thus, let $\boldsymbol{\Psi} = [\boldsymbol{\Psi}^c, \boldsymbol{\Psi}^v, \boldsymbol{\Psi}^{l_{s,1}}, \boldsymbol{\Psi}^{l_{s,2}}, \dots, \boldsymbol{\Psi}^{l_{s,K}}]$, where $\Psi^{l_{s,k}}$ is the two-dimensional matrix of size $\sum_{l_s}^{n} P(l_s) \times n_{states}$ storing the number of components of type k working in state s or above. Then, the extended formula of the concept of DAS for the consideration of systems with multiple components types can be expressed as

$$R(s,t) \geq \sum_{j=1}^{\sum_{l_s}^{n} P(l_s)} \Psi^c(j,s) \prod_{k=1}^{K} \left[R_{x_k}(\Psi^v(j,s) \mid t) - R_{x_k}(s \mid t)\right]^{n_k - \Psi^{l_{s,k}}(j,s)} \left[R_{x_k}(s \mid t)\right]^{\Psi^{l_{s,k}}(j,s)}. \quad (8)$$

Similar to Equation (7), this inequality can be vectorized when implemented. This significantly increases the computational efficiency. However, combinatorial theory suggests that the computational cost will grow exponentially with increasing number of components n. Consequently, aggregation strategies should be identified in order to restrain the computational cost.

2.4. Aggregation Strategies for the Condensation of the Diagonal Approximated Signature Psi

Some reasonable aggregation strategy should be applied to $\boldsymbol{\Psi}$ to restrain the computational cost when it comes to systems with a large number of comprised components $n > 8$. Thereby, the sum over $\sum_{l_s}^{n} P(l_s)$ should be reduced to a sum over J, where $J \ll \sum_{l_s}^{n} P(l_s)$. Then, an underapproximation of the CSSF for a system with multiple component types is obtained with increasing computational efficiency when following the formula given as

$$R(s,t) \geq \sum_{j=1}^{J} \Psi^c(j,s) \prod_{k=1}^{K} \left[R_{x_k}(\Psi^v(j,s) \mid t) - R_{x_k}(s \mid t)\right]^{n_k - \Psi^{l_{s,k}}(j,s)} \left[R_{x_k}(s \mid t)\right]^{\Psi^{l_{s,k}}(j,s)}. \quad (9)$$

Similar to the concept of survival signature, the DAS can be condensed in terms of $l_{s,k}$, the number of components of type k working in state s or above, for $k = 1,2,\dots,K$. In order to condense $\boldsymbol{\Psi}$, find the minimum values of $\Phi(s, l_s, p)$ of all permutation $p = 1,2,\dots,P(l_s)$ for all possible combinations of the numbers of components of type k working in state s or above that fulfill the condition $\sum_{k=1}^{K} l_{s,k} = l_s$. Thereby, $\Psi^c(j,s)$ comprises the number of according permutations of the component state vector that were joined together. Note that storing only one minimum value is the cheapest aggregation strategy in terms of computational cost. However, this will also generate the largest CSSF error margin in terms of its volume over the entire domain. Consequently, choosing a reasonable aggregation strategy always comes along with the tradeoff between computational cost and a minimum error margin.

3. Case Studies

As basis for the analyses, consider a system that has a coherent and a diagonally state positive structure function mapping continuous-state component performances to the continuous-state system performance. As in (Winnewisser et al., 2023), such a structure function can be established for an infrastructure network, where the system topology is modeled as a graph and the network efficiency is considered as the performance of the system. Route travel times are assigned as edge weights between nodes representing crossways and cities and monotone-increasing stochastic road degradation affects the travel times along edges. For the current case studies, simpler structure functions were investigated. However, these share the same properties as the more realistic infrastructure models. The fundamental form of the coherent structure function is given as

$$\phi(\boldsymbol{x}) = \sum_{i=1}^{n_{considered}} x_i^{\frac{1}{c_i}}, \quad (10)$$

$\boldsymbol{x} \in [0,1]^{n_{considered}}$ and $\boldsymbol{c} \in \mathbb{N}_{>0}^{n_{considered}}$. The both case studies will differ in terms of $\boldsymbol{c}$ that is a $n_{considered}$-dimensional parameter vector comprising positive natural numbers. The component degradation was assumed to be modeled by a family of Beta distributions depending on the according parameters, assured to be monotonically decreasing over time. Two component types were assumed. The corresponding time-dependent α_k and β_k parameters dependent on the component type $k = 1,2$ and are determined as $\alpha_1(t) = 5 - 0.35t$, $\beta_k(t) = 0.11 + 0.9t$, $\alpha_2(t) = 6 - 0.4t$, $\beta_2(t) = 0.51 + 0.5t$.

3.1. Case Study 1: Computation Time Study

First, consider the defined structure function with $\boldsymbol{c}$, where $c_i = n_{considered} + 1$ for $i = 1,2,\dots,n_{considered}$. A study was conducted investigating the required computation times for evaluating the entire CSSF or an underapproximation of it for $n_{considered} = 2,3,\dots,14$. The CSSF is estimated by means of MCS and approximated via the concept of DAS represented via Ψ (DASP) and the condensed representation of the DASP (DASPC), compare formulas 1,2,3. Component types are assigned as $\boldsymbol{t}(i) = [2, 2, 2, 1, 2, 1, 2, 2, 2, 1, 2, 2, 1, 1]$ for $i = 1,2,\dots,n_{considered}$. The CSSF was evaluated for equidistant $n_{states} = 101$ in $[0,1]$ and equidistant $n_{timesteps} = 101$ in the interval $[0,10]$.

The results of the obtained CSSFs are illustrated in Fig. 3 and Fig. 4 for $n_{considered} = 14$. An estimate of the true solution obtained by MCS with 10 000 samples for each state s and time step t is shown in Fig. 3 a). Slight variations can be observed for all contour lines but particularly for the contour line $c(s = 1)$. The underapproximation of the CSSF computed by means of DASP is illustrated in Fig. 3 b). A significant CSSF error

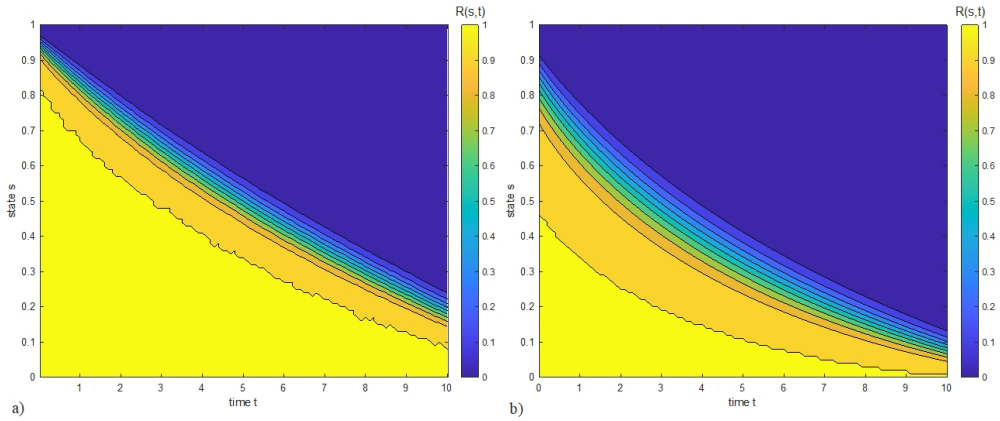

Fig. 3. Case Study 1: (a) CSSF via MCS; (b) CSSF via DASP.

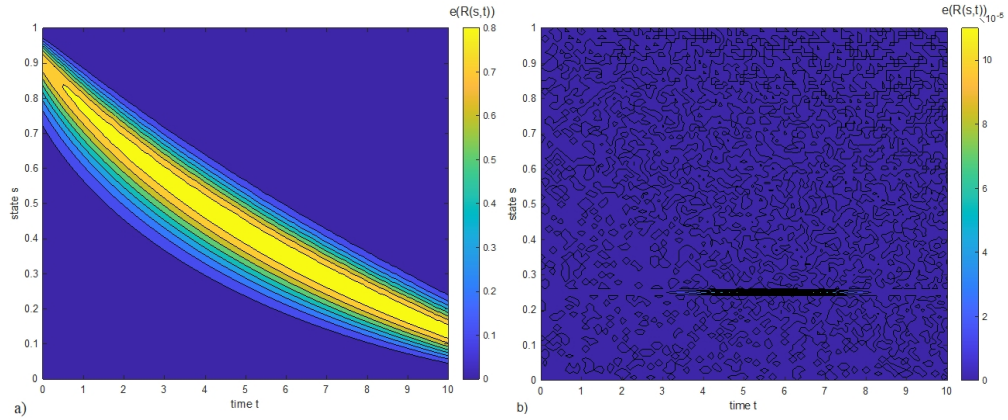

Fig. 4. Case Study 1: (a) Error between CSSF (MCS) and CSSF (DASP); (b) Error between CSSF (DASP) and CSSF (DASPC).

margin can be observed when considering the error between the CSSF from MCS and DASP in Fig. 4 a). Note that the error margin can be further reduced with accordingly adapted formulas, e.g., defined in a recursive manner. Further, the considered system is diagonally symmetric. Therefore, $\Phi(s, l_s, p) = \Phi(s, l_s, q)$ for $p = 1,2,\dots,P(l_s)$ and $q = 1,2,\dots,P(l_s)$. Consequently, all values stored with DASP are preserved in DASPC after applying the proposed condensation strategy and the corresponding error between DASP and DASPC is mainly in the magnitude of machine precision.

The results of the computation time study are shown in Fig. 5. 10 samples were used for each $n_{considered}$ to estimate the mean computation time. For the approach based on MCS, a linear relation can be observed. The computations have been vectorisaed as well. The slope of this linear dependency solely depends on the computational cost of the structure function, as it has to be evaluated N_{MCS} times. The structure function considered in these case studies can be considered as cheap compared to more complicated models as outlined in the previous section.

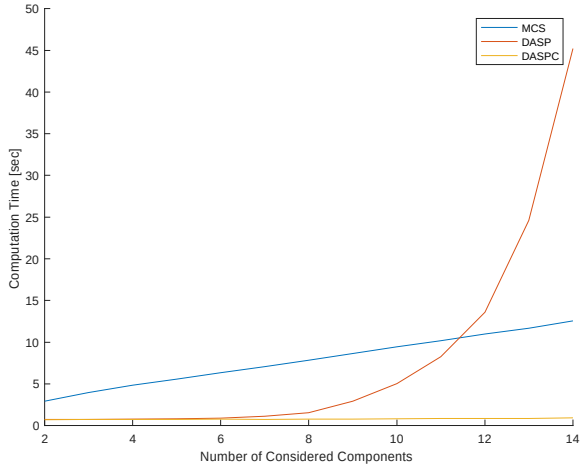

Fig. 5. Case Study 1: Computation time study for MCS, DASP, DASPC in terms of an increasing number of components *n*.

The DASP shows exponential growth of the computation time for an increasing number of components $n_{considered}$. For $n_{considered} = 12$, the computation of an underapproximation of the CSSF exceeds the time required for estimating the true CSSF via MCS. In contrast, the computation time required by the DASPC increases over the range of $n_{considered}$ marginally. Table 1 shows the corresponding numerical values. Note that the computational cost of the DASP and DASPC do not increase with increasing computational cost of the structure function but depend on the according aggregation strategy and the resulting size of the three-dimensional matrix DASPC.

Table 1. Computation time study – numerical values [sec].

$n_{considered}$/ Approach	2	3	4	5	6	7	8	9	10	11	12	13	14
MCS	3.02	4.13	5.09	5.89	6.63	7.49	8.22	9.06	9.83	10.62	11.75	12.09	12.86
DASP	0.70	0.72	0.77	0.84	0.92	1.11	1.53	2.93	5.04	8.15	13.68	24.70	45.71
DASPC	0.67	0.67	0.70	0.72	0.73	0.74	0.75	0.75	0.78	0.80	0.81	0.86	0.88

3.2. Case Study 2: Increased CSSF Error Margin Due to Aggregation

As it can be seen in Fig. 5., it is of utmost importance to condense the DASP in order to be applicable to systems with a large number of components due to high computational cost. However, the volume of the overall CSSF error margin between DASP and DASPC strongly depends on the properties of the underlying coherent structure function $\phi(\boldsymbol{x})$. An appropriate condensation strategy should be identified to balance the tradeoff between computational cost and a minimum CSSF error margin. The aggregation strategy proposed in Section 3.4. was applied for this case study.

In this example, the structure function is unsymmetric due to the assignment of $\boldsymbol{c}(i)$ as $c_i = n_{considered} + i$ for $i = 1,2, \dots, n_{considered}$, compare Equation 10. Assign $n_{considered} = 16$ and $\boldsymbol{t}(i) = [2, 2, 2, 1, 2, 1, 2, 2, 2, 2, 1, 2, 1, 2, 1, 2]$ for $i = 1,2, \dots, n_{considered}$. Fig. 6. shows the obtained results by means of MCS and DASP, while Fig. 7. illustrates the error margins between the CSSFs computed via MCS and DASP as well as DASP and DASPC. The estimation of the true CSSF computed by means of MCS is given in Fig. 6. a). In

comparison, the DASP enables to compute an underestimation that is shown in Fig. 6. b) and the corresponding error margin in Fig. 7. a). Similar to the first case study, the volume of the error margin is significant. Fig. 7. b) shows the CSSF error margin between DASP and DASPC. This CSSF error margin is noticeable but small compared to the one shown in Fig. 7. a).

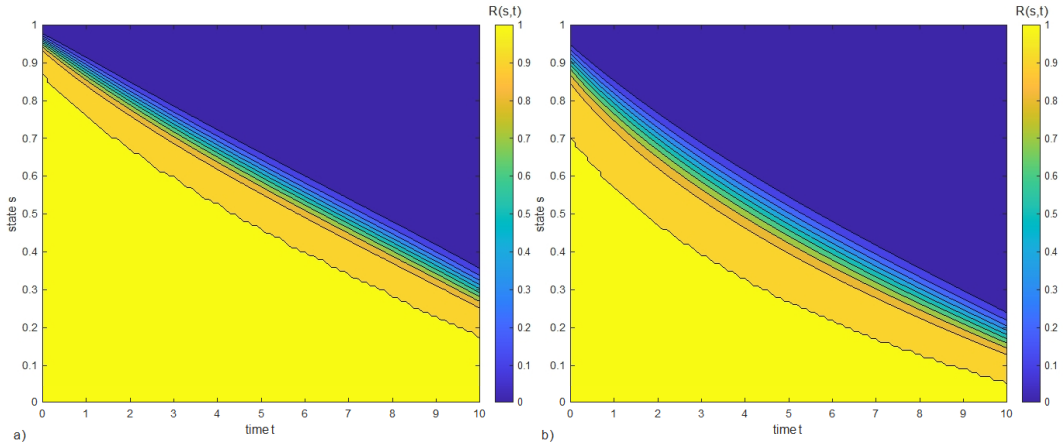

Fig. 6. Case Study 2: (a) CSSF via MCS; (b) CSSF via DASP.

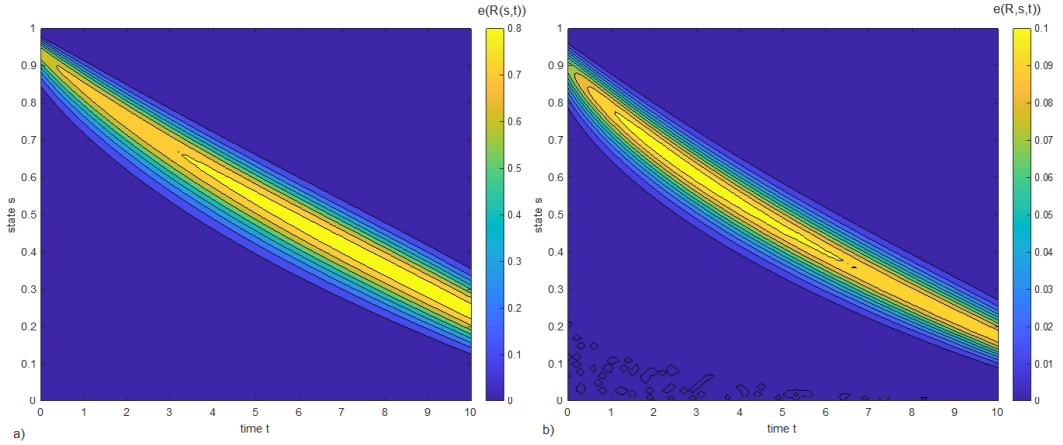

Fig. 7. Case Study 2: (a) Error between CSSF (MCS) and CSSF (DASP); (b) Error between CSSF (DASP) and CSSF (DASPC).

4. Conclusions and Outlook

In the current work, the concept of DAS, originally introduced in (Winnewisser et al., 2023), is extended to consider systems that comprise components of multiple types. Further, a numerically efficient representation of the DAS was introduced and is referred to as DASP. Based on this representation the computations can be vectorised, leading to high computational efficiency when the number of comprised components $n \leq 6$. To overcome this limitation a basic aggregation strategy is proposed. The application of such generates the condensed DASP, referred to as DASPC, that results in tremendously reduced computation cost for any number of components n compared to MCS and DASP. The computation time study illustrated in the case studies proofs this advantageous behavior of the DASPC. In contrast to the MCS, the computational cost of the concept of DASP(C) in repeated model evaluations is independent of the computational cost of the structure function and does not require sampling. It only depends on the number of considered components and component types and the

aggregation strategy to generate DASPC. A tradeoff has to be made between reducing computational cost and reducing errors between the generated CSSFs.

An estimate of the true CSSF is obtained by means of MCS. The error between the true CSSF and its MCS estimate depends on the number of samples N_{MCS}. The concept of DASP(C) evaluates an underapproximation of the CSSF based on explicit formulas. For the basic formulas proposed in this work a significant underapproximation can be observed. Nevertheless, the concept of DAS(PC) allows a more comprehensive analysis than other concepts of signatures if the underlying probability distributions characterise the random continuous performance states.

The proposed formulas should be generalised to be also applicable to diagonally state negative systems. Further, more sophisticated aggregation strategies can be established to achieve an optimal tradeoff between computational cost and accuracy. In addition, combining the approach with further surrogate models, e.g., based on artificial intelligence facilitates the application to large scale systems, cf. (Shi et al., 2023). Similarly, the amalgamation with estimation procedures, such as proposed in (Behrensdorf et al., 2021), should enable this as well. One of the focal points of future developments will be the refinement of the formulas based on recursion for reducing the CSSF error margin between the true solution CSSF and the underapproximation obtained by means of the DAS concept.

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