Advances in Reliability, Safety and Security, Part 2 – Kołowrocki, Magryta-Mut (eds.)
© 2024 Polish Safety and Reliability Association, Gdynia, ISBN 978-83-68136-14-2 (printed), ISBN 978-83-68136-01-2 (electronic)

Advances in Reliability, Safety and Security

ESREL 2024
Monograph Book Series

Maintenance Of Degrading Units: New Policy And Some Critical Considerations About Value Of Inspections

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Abstract

In this paper, a new maintenance policy is proposed for deteriorating units whose degradation paths are affected by unit-to-unit variability. The degradation process is modeled by using a gamma process with random effect. The units are assumed to fail when their degradation level exceeds a (fixed) given threshold. It is assumed that failures are not self-announcing and (consequently) that only an inspection can allow to say with certainty if a unit is failed or not. The maintenance policy is defined by assuming that a unit can be subjected to no more than two inspections. The first inspection is performed at a planned time. Hence, based on the result of the first inspection the unit can be immediately replaced, a second inspection time can be planned, or a future replacement time can be possibly defined. In the latter case, the unit will be replaced at the future replacement time, independently of its state, without performing a second inspection. Differently, when the decision consists in executing a second inspection, at the second inspection time, once again, based on the result of the inspection, the unit can be immediately replaced or its replacement can be postponed to a given successive time, where it will be replaced, independently of its state, without performing a third inspection. After each replacement, the unit is considered as good as new. The results obtained by using the proposed policy are compared with those obtained by using a similar existing (simplified) policy that includes only one inspection and a classical age-based policy, which does not make use of any inspection. The comparison is performed by considering different values of logistic and inspection costs. Obtained results show that depending on the scenario either of the considered policies can be preferred to the others. Some considerations about the value of information gained by performing an inspection close the paper.

Keywords: long-run average cost rate, condition-based maintenance, age-based maintenance, gamma process, random effect

1. Introduction

This paper deals with maintenance of deteriorating units that are assumed to fail when their degradation level exceeds an assigned failure threshold. Maintenance strategies of units subjected to this kind of failure are customarily developed by adopting either an age-based or a condition-based approach (e.g., see (Ahmad and Kamaruddin, 2012) and (Alaswad and Xiang, 2017)). Optimal policies are usually identified by optimizing appropriate objective functions (e.g., see (Wang and Pham, 2006), (Gertsbakh, 2013), (Finkelstein et al., 2016), and (Cha et al. 2017)). Here, we focus on the minimization of the long-run average maintenance cost rate.

In general, condition-based maintenance (CBM) strategies tend to outperform age-based maintenance (ABM) strategies, allowing (at the same time) to reduce failures and to use units for a greater portion of their useful life (see (de Jonge et al., 2017) for a critical discussion of the relative advantages of CBM over ABM).

A possible defect of many existing CBM policies is that they assume that preventive replacements are only permitted at inspection epochs (e.g., see (Wang and Pham, 2013) and (Huynh et al., 2019).

To overcome this potential limitation, (Esposito et al., 2022) have recently proposed a one-inspection maintenance policy that breaks this scheme, where based on the information gained by performing the inspection is decided whether to immediately replace the unit or to postpone its replacement to a future time, where it is replaced without performing a second inspection. The rationale behind this strategy is that, especially in cases where inspection costs are very high, it is reasonable to presume that maintenance policies that allow to avoid costly inspections that are likely to lead to foregone decisions could give economic advantages (e.g., see (Fauriat and Zio, 2020), (Yuan et al., 2021), and (Kim et al., 2022)).

Similar ideas are also proposed, with different motivations and/or under different settings, in (Crowder and Lawless, 2007), (Finkelstein et al. 2020), (Cha et al., 2021), and (Cha et al., 2022). In particular, (Crowder and Lawless, 2007), after discussing in detail the case where the unit is subjected to a single inspection, envision the possibility of performing a second inspection at the future scheduled replacement time, instead of performing an automatic replacement.

Here, inspired by (Crowder and Lawless, 2007), we propose a new hybrid maintenance policy that extends a special case of the adaptive one-inspection policy presented in (Esposito et al., 2022) by including the possibility of carrying out a second inspection.

In fact, this new maintenance policy assumes that a unit can be subjected to one or two inspections, according to convenience. More specifically, at the first inspection time, based on the result of the inspection, it is possible to immediately replace the unit, to plan a second inspection at a future time, or to postpone its replacement to a possibly different future time. In this latter case, the unit will be replaced at the future time without performing a second inspection. Differently, when the unit is subjected to the second inspection, based on the result of the first and second inspections, it will be possible to either immediately replace the unit or to further postpone its replacement to a given subsequent time, where the unit will be systematically replaced, without performing a third inspection.

The proposed maintenance policy is applied by assuming that the degradation phenomenon of interest can be suitably described by the gamma process with random effect firstly proposed in (Lawless and Crowder, 2004). Under this well-known model the growth of the degradation level of the generic unit is modeled by a gamma process. The presence of unit-to-unit variability is modeled by assuming that the scale parameter varies randomly (from unit-to-unit) according to a gamma random variable. The marginal process is Markovian, with non-independent and non-stationary increments. The fact that increments are non-stationary makes the optimization task more interesting from the pure mathematical point of view, with respect to classical studies on maintenance policy for gamma degrading units, where the degradation process is described by using a homogeneous gamma process (e.g., see (van Noortwijk, 2009) and (Alaswad and Xiang, 2017) for a comprehensive review of existing literature). On the other hand, at the same time, the Markov property allows to contain the complexity of the study, by making the decision at any inspection time dependent only on the current age and state of the unit.

However, apart from these computational aspects, the adoption of the gamma process with random effect, here is mainly motivated by the fact that its use allows boosting the value of the inspections. In fact, under this model, even a single inspection allows to obtain precious information about the unobservable value of the unit-specific (latent) scale parameter, which distinguishes weak units from strong ones, because the rapidity with which the degradation level progresses over time depends on its value.

As already mentioned above, the failure is supposed to occur when the degradation level exceeds an assigned failure threshold. Hence, the lifetime of a unit is assumed to coincide with the first passage time of its degradation process to the failure threshold. More in particular, we assume that the considered soft failure is not self-announcing so that only an inspection can allow to say with certainty if a unit is failed. Accordingly, we also suppose that a failed unit could continue to operate, albeit with reduced performance and/or additional cost and that both corrective and preventive replacements can be performed only at the predetermined inspection/replacement time. Finally, we assume that after each replacement, the unit is considered as good as new. Hence, time elapsing between two successive replacements is seen as a renewal cycle. Both replacement times and inspection times are decision variables that should be set a priori.

Beyond proposing a new maintenance policy, the main aim of the paper is to investigate the value of the information that can be gathered by performing inspections and thus evaluating the opportunity/convenience of performing one or more inspections in cases where inspection costs are very high.

To this aim, results obtained by adopting the proposed two-inspection policy is compared both to those obtained by using a comparable (simplified) version of the one-inspection policy suggested in (Esposito et al., 2022) and to those obtained under a classical (zero-inspection) age-based policy.

The comparison is developed by considering three different realistic experimental scenarios that differ for inspection and logistic costs and include a case where these costs are very high (e.g., see (Alaswad and Xiang, 2017)), a case where they are moderate, and a case where they are relatively low. Aim of the study is to show that either of the considered maintenance policies can be preferred to the others depending on the experimental scenario.

The paper is structured as follows: in Section 2 a brief description of the adopted stochastic degradation model is provided. Section 3 and 4 describe the proposed two-inspection hybrid maintenance policy and the cost model, respectively. Section 5 is devoted to the formulation of the long-run average maintenance cost rate. Section 6 is devoted to the application. A conclusion section closes the paper.

2. The gamma degradation process with random effect

The evolution over time of the degradation level of the considered unit is described by using the gamma process with random effect firstly suggested in (Lawless and Crowder, 2004). This model allows to account for the presence of forms of heterogeneity between the degradation paths of gamma degrading units that are nominally identical, when said differences cannot be explained by incorporating covariates into the basic gamma process.

Under this model, the degradation process of the units pertaining to the population of interest is described by using a gamma process that has an assigned age function and scale parameter that varies randomly from unit to unit according to a gamma probability distribution. More specifically, the conditional pdf of the degradation increment $\Delta W(t,\ t+\Delta t) = W(t+\Delta t) - W(t)$, given the value λ of the random scale parameter Λ, is expressed as:

$$f_{\Delta W(t,t+\Delta t)|\Lambda}(\delta|\lambda) = \frac{\lambda^{\Delta\eta(t,t+\Delta t)} \cdot \delta^{\Delta\eta(t,t+\Delta t)-1}}{\Gamma(\Delta\eta(t,t+\Delta t))} \cdot e^{-\lambda\cdot\delta}, \delta \geq 0 \tag{1}$$

and the scale parameter Λ is assumed to have the gamma pdf:

$$f_\Lambda(\lambda) = \frac{c^d \cdot \lambda^{d-1}}{\Gamma(d)} \cdot e^{-c\cdot\lambda},\ \ \lambda > 0, \tag{2}$$

where $\Gamma(y) = \int_0^\infty u^{y-1} \cdot e^{-u} \cdot \mathrm{d}u$ denotes the complete gamma function, $\eta(t)$ is a non-negative monotonic increasing function, usually referred to as the age function, $\eta(t,\ t+\Delta t) = \eta(t+\Delta t) - \eta(t)$, c and d $(c, d > 0)$ are the scale and shape and parameters of the pdf (2), respectively. In this paper, the age function is modeled as $\eta(t) = a \cdot t^b$. Moreover, it is assumed that $W(0) = 0$.
From (1) and (2), it results that the degradation level of the unit at t, $W(t)$, has marginal pdf:

$$f_{W(t)}(w) = \frac{c^d \cdot w^{\eta(t)-1}}{B(\eta(t),d)\cdot(w+c)^{\eta(t)+d}},\ \ w \geq 0, \tag{3}$$

marginal cdf:

$$F_{W(t)}(w) = \mathbb{B}\left(\frac{w}{w+c};\ \eta(t), d\right),\ \ w \geq 0,$$

mean (that exists for $d > 1$):

$$E\{W(t)\} = \frac{c\cdot\eta(t)}{d-1},$$

and variance (that exists for $d > 2$):

$$V\{W(t)\} = \frac{c^2\cdot\eta(t)\cdot[\eta(t)+d-1]}{(d-1)^2\cdot(d-2)},$$

where:

$$\mathbb{B}(z;\alpha,\beta) = \frac{\int_0^z u^{\alpha-1}\cdot(1-u)^{\beta-1}\cdot \mathrm{d}u}{B(\alpha,\beta)},\ \ \alpha,\beta > 0 \tag{4}$$

is the regularized beta function and

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

is the beta function.

Moreover, it is also possible to prove (e.g., see (Esposito et al., 2023)), that the (marginal) degradation process $\{W(t); t\geq 0\}$, is Markovian and that the degradation increment $\Delta W(t,\ t+\Delta t)$, given $W(t) = w$, has conditional pdf:

$$f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w) = \frac{1}{B(\Delta\eta(t,t+\Delta t),\eta(t)+d)} \frac{(w+c)^{\eta(t)+d} \cdot \delta^{\Delta\eta(t,t+\Delta t)-1}}{(w+\delta+c)^{\eta(t+\Delta t)+d}},\ \ \delta \geq 0, \tag{5}$$

and conditional cdf:

$$F_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w) = \mathbb{B}\left(\frac{\delta}{\delta+w+c};\ \Delta\eta(t,t+\Delta t), \eta(t)+d\right),\ \delta \geq 0. \tag{6}$$

As already mentioned, we assume that a unit fails when its degradation level exceeds an assigned failure threshold, hereinafter indicated by w_M. Thus, given that the process $\{W(t), t \geq 0\}$ is monotone increasing, the useful life X of the unit (i.e., its time to failure):

$$X = \inf\{x: W(x) > w_M\}$$

can be defined as the first and sole passage time of $\{W(t), t \geq 0\}$ to the failure threshold w_M.

In the rest of this section, we report some results, which involve the lifetime X, that have been used to formulate the expected cost (10).

From the conditional cdf (6), denoting by τ_a and τ_b two generic reference time, such that $0 < \tau_a < \tau_b$, it is possible to readily obtain the conditional cdf of X given $W(\tau_b) = w_b$ and $W(\tau_a) = w_a$ in the cases where $w_a \leq w_b \leq w_M$. In fact, this cdf can be expressed as:

$$F_{X|W(\tau_b),W(\tau_a)}(x|w_b, w_a) = F_{X|W(\tau_b)}(x|w_b) = P(X \leq x|W(\tau_b) = w_b)$$

$$= P(W(x) > w_M|W(\tau_b) = w_b) = \begin{cases} 0, & w_b \leq w_\mathrm{M},\ x \leq \tau_b \\ 1 - \mathbb{B}\left(\frac{w_M - w_b}{w_M + c};\ \Delta\eta(\tau_b, x), \eta(\tau_b) + d\right), & w_b \leq w_\mathrm{M},\ x > \tau_b \end{cases} \tag{7}$$

where the first equality is justified by the fact that the process is Markovian and the second equality can be explained by observing that, since the process $\{W(t), t \geq 0\}$ is monotone increasing, the event $\{X \leq x\}$ is equivalent to the event $\{W(x) > w_\mathrm{M}\}$.

Differently, in the case where $w_a < w_M < w_b$ the conditional cdf of X given $W(\tau_a) = w_a$ and $W(\tau_b) = w_b$ can be expressed as in (8):

$$F_{X|W(\tau_a),W(\tau_b)}(x|w_a, w_b) = P(X \leq x|W(\tau_b) = w_b, W(\tau_a) = w_a)$$

$$= P(W(x) > w_M|W(\tau_b) = w_b, W(\tau_a) = w_b) =$$

$$= \begin{cases} 0, & w_a < w_M < w_b, \quad x \leq \tau_a \\ 1 - \mathbb{B}\left(\frac{w_M - w_a}{w_b - w_a}; \eta(x) - \eta(\tau_a), \eta(\tau_b) - \eta(x)\right), & w_a < w_M < w_b, \quad \tau_a < x < \tau_b \\ 1, & w_a < w_M < w_b, \quad x > \tau_b. \end{cases} \tag{8}$$

In fact, being:

$$P(W(x) > w_M|W(\tau_b) = w_b, W(\tau_a) = w_a) = 1 - \int_{w_a}^{w_M} \frac{f_{W(\tau_b)|W(x)}(w_b|w) \cdot f_{W(x)|W(\tau_a)}(w|w_a)}{f_{W(\tau_b)|W(\tau_a)}(w_b|w_a)} \cdot \mathrm{d}w,$$

from (3), (5), and (4) we have:

$$F_{W(x)|W(\tau_a),W(\tau_b)}(x|w_a, w_b) = 1 - \frac{1}{\mathrm{B}(\eta(x) - \eta(\tau_a), \eta(\tau_b) - \eta(x))} \cdot \int_0^{\frac{w_M - w_a}{w_b - w_a}} y^{\eta(x) - \eta(\tau_a) - 1} \cdot (1 - y)^{\eta(\tau_b) - \eta(x) - 1} \cdot \mathrm{d}y$$

$$= 1 - \mathbb{B}\left(\frac{w_M - w_a}{w_b - w_a}; \eta(x) - \eta(\tau_a), \eta(\tau_b) - \eta(x)\right),$$

that coincides with the cdf (8).

Note that, for the sake of economy of notation, the cdfs in (7) and (8) have been indicated by using the same symbol. However, as it is explicitly specified in the text, the (7) should be used if and only if $w_b \leq w_\mathrm{M}$ and the (8) if and only if $w_a < w_M < w_b$.

From (8), when $w_a < w_M < w_b$, the conditional mean of X, given $W(\tau_b) = w_b$ and $W(\tau_a) = w_a$, can be expressed as:

$$E\{X|W(\tau_b) = w_b, W(\tau_a) = w_a\} = \int_{\tau_a}^{\tau_b} x \cdot f_{X|W(\tau_b),W(\tau_a)}(x|w_b, w_a) \cdot \mathrm{d}x$$

$$= \tau_a + \int_{\tau_a}^{\tau_b} \mathbb{B}\left(\frac{w_M - w_a}{w_b - w_a}; \eta(x) - \eta(\tau_a), \eta(\tau_b) - \eta(x)\right) \cdot \mathrm{d}x.$$

Finally, from (7), when $w_b \leq w_\mathrm{M}$, given $W(\tau_b) = w_b$ and $W(\tau_a) = w_a$, the conditional mean of the variable $g(X)$ defined by the following transformation:

$$g(X) = \begin{cases} X, & X \leq \tau_c \\ 0, & X > \tau_c \end{cases}$$

with $\tau_c \geq \tau_b$, can be computed as:

$$E\{g(X)|W(\tau_b) = w_b, W(\tau_a) = w_a\} = \int_{\tau_b}^{\tau_c} x \cdot f_{X|W(\tau_b),W(\tau_a)}(x|w_b, w_a) \cdot \mathrm{d}x$$

$$= \tau_b - \tau_c \cdot \mathbb{B}\left(\frac{w_M - w_b}{w_M + c}; \eta(\tau_c) - \eta(\tau_b); \eta(\tau_b) + d\right) + \int_{\tau_b}^{\tau_c} \mathbb{B}\left(\frac{w_M - w_b}{w_M + c}; \eta(x) - \eta(\tau_b); \eta(\tau_b) + d\right) \cdot \mathrm{d}x.$$

It is worth to remark that, under the considered setting, the marginal process $\{W(t), t \geq 0\}$ is age and state dependent, a feature that the process preserves also when (as it is assumed in Section 4) the temporal variability of the unit-specific paths are described by homogeneous gamma processes (e.g., see (Giorgio and Pulcini, 2018)). The dependence on the state is generated by the presence in the population of units whose degradation levels evolve over time with different rapidity (i.e., the future degradation growth of units whose degradation level is predicted to be high because these units are suspected to be weak). In fact, it is exactly the existence (and the understanding of the nature) of this dependency that allows to distinguish the weak units from the strong ones.

From the practical point of view, given that $\{W(t), t \geq 0\}$ is age and state dependent, the probability that a failure occurs in a future interval, which under the basic homogenous gamma process (e.g., (Grall et al., 2002), (Castanier et al., 2003), (Castanier et al., 2005), and (Huynh et al., 2019)) depends only on the length of the time interval and on the gap existing between the current degradation level of the unit and the failure threshold, here also directly depends on the current degradation level and age of the unit.

3. The maintenance model

We consider a heterogeneous population of degrading units where the degradation growth can be described by the gamma process with random effect presented in Section 2. Maintenance activities refer to a generic unit randomly selected from the considered population. The proposed policy is developed based on the following assumptions:

- maximum two inspections can be performed during the lifetime of the unit;
- inspections are instantaneous and non-destructive;
- failures are not self-announcing;
- a failed unit can continue to operate with reduced performance, causing additional costs;
- both corrective and preventive replacements restore the unit to an "as good as new" state.

The suggested policy consists in performing a first inspection at time τ_1 where it is decided whether to immediately replace the unit, to postpone its replacement to a future time τ_2 (i.e., $\tau_2 > \tau_1$), or to plan a second inspection at a future time τ_3 (i.e., $\tau_3 > \tau_1$). If the replacement is postponed at time τ_2, no other inspection will be performed and the unit will be replaced at τ_2 irrespective of its state. Differently, in the case a second inspection is performed, at time τ_3 it will be decided whether to immediately replace the unit or to postpone its replacement to a further future time τ_4 (i.e., $\tau_4 > \tau_3$). In this latter case, at τ_4 the unit will be replaced without performing any other inspection. All decisions are made according to condition-based rules. Inspections consist in measuring the degradation level of the unit. It is supposed that the adopted measurement procedure allows to observe the exact value of the degradation state of the unit.

Considered that failures are not self-announcing (i.e., given that failed units do not reveal their state), it is assumed that replacements are executed only at the predetermined inspection/replacement times.

The condition-based rule adopted to make the decision is described in Table 1.

Hereinafter, w_i indicates the value of $W(\tau)$ at τ_i $(i = 1,2,3,4)$ and l_1, L_1, and L_3 are (maintenance) threshold levels. It is $0 \leq l_1 \leq L_2 \leq w_M$ and $0 \leq L_3 \leq w_M$.

Table 1. Condition-based rules.

Inspection at τ_1		Inspection at τ_3	
Result	Decision	Result	Decision
$w_1 > L_1$	Replacement at τ_1		
$l_1 < w_1 \le L_1$	Replacement at τ_2		
$w_1 \le l_1$	Inspection at τ_3	$w_3 > L_3$	Replacement at τ_3
		$w_3 \le L_3$	Replacement at τ_4

Table 2 summarizes all possible experimental scenarios and reports for each of them the maintenance action to be taken.

Table 2. Maintenance actions.

Inspection at τ_1			Inspection at τ_3			
Result	State at τ_1	State at τ_2	Result	State at τ_3	State at τ_4	Maintenance action
$w_1 > L_1$	$w_1 > w_M$					Corrective at τ_1
	$w_1 \le w_M$					Preventive at τ_1
$l_1 < w_1 \le L_1$		$w_2 > w_M$				Corrective at τ_2
		$w_2 \le w_M$				Preventive at τ_2
$w_1 \le l_1$			$w_3 > L_3$	$w_3 > w_M$		Corrective at τ_3
				$w_3 \le w_M$		Preventive at τ_3
			$w_3 \le L_3$		$w_4 > w_M$	Corrective at τ_4
					$w_4 \le w_M$	Preventive at τ_4

Table 3 gives for each maintenance action details about the useful life and the length of a maintenance cycle, denoted by X and $T(action)$, respectively.

Table 3. Useful life and length of a maintenance cycle for each maintenance action.

Maintenance action	X	$T(action)$
Corrective at τ_1	$X \le \tau_1$	τ_1
Preventive at τ_1	$X > \tau_1$	τ_1
Corrective at τ_2	$\tau_1 < X \le \tau_2$	τ_2
Preventive at τ_2	$X > \tau_2$	τ_2
Corrective at τ_3	$\tau_1 < X \le \tau_3$	τ_3
Preventive at τ_3	$X > \tau_3$	τ_3
Corrective at τ_4	$\tau_3 < X \le \tau_4$	τ_4
Preventive at τ_4	$X > \tau_4$	τ_4

The times τ_1, τ_2, τ_3, τ_4 and the threshold limits l_1, L_1, and L_3 should be intended as design parameters of the considered policy. In this paper, these parameters are set with the aim of minimizing the long-run average maintenance cost rate. Hereinafter, we will indicate by $\boldsymbol{\zeta}$ the vector of design parameters (i.e., $\boldsymbol{\zeta} = \{\tau_1, \tau_2, \tau_3, \tau_4, l_1, L_1, L_3\}$) and by $\boldsymbol{\zeta}^*$ the value of $\boldsymbol{\zeta}$ that defines the optimal policy.

The cost model is formulated as shown in Table 4. Costs are defined for each maintenance action described in Table 3, as a function of the useful life X, when this is necessary.

In Table 4, c_c is the cost of a corrective replacement, c_p is the cost of a preventive replacement, c_i is the inspection cost, c_l is the logistic cost (incurred each time a maintenance action is taken), and c_d is the downtime cost rate per unit of time caused by operating the unit in a failed state. The total downtime cost in a maintenance cycle is computed as the product of the downtime and this fixed cost rate. The downtime is assumed to coincide with the time elapsing from the failure occurrence to the time at which the unit is replaced. Finally, $C(\text{action}, X)$ is the maintenance cost per cycle formulated, scenario by scenario, for given values of the arguments action and X.

It is worth to emphasize that in Table 4, in agreement with the assumption that failures are not self-announcing, X is always denoted with the capital letter to intend that the lifetime of the unit should be regarded as a random variable, even when the degradation level at the inspection time is known.

Table 4. Maintenance costs for each maintenance action as a function of the lifetime X.

Maintenance action	X	$C(\text{action}, X)$
Corrective at τ_1	$X \leq \tau_1$	$c_l + c_i + c_c + c_d \cdot (\tau_1 - X)$
Preventive at τ_1	$X > \tau_1$	$c_l + c_i + c_p$
Corrective at τ_2	$\tau_1 < X \leq \tau_2$	$2 \cdot c_l + c_i + c_c + c_d \cdot (\tau_2 - X)$
Preventive at τ_2	$X > \tau_2$	$2 \cdot c_l + c_i + c_p$
Corrective at τ_3	$\tau_1 < X \leq \tau_3$	$2 \cdot c_l + 2 \cdot c_i + c_c + c_d \cdot (\tau_3 - X)$
Preventive at τ_3	$X > \tau_3$	$2 \cdot c_l + 2 \cdot c_i + c_p$
Corrective at τ_4	$\tau_3 < X \leq \tau_4$	$3 \cdot c_l + 2 \cdot c_i + c_c + c_{d\cdot} \cdot (\tau_4 - X)$
Preventive at τ_4	$X > \tau_4$	$3 \cdot c_l + 2 \cdot c_i + c_p$

Finally, by using the renewal/reward theorem (e.g., see (Ross, 1983)), the long-run average maintenance cost rate $C_\infty(\boldsymbol{\zeta})$ can be formulated as:

$$C_\infty(\boldsymbol{\zeta}) = \frac{E\{C(action, X)\}}{E\{T(\text{action})\}} \tag{9}$$

where the expectations have to be taken with respect to $W(\tau_1)$, $W(\tau_3)$, and X. In this respect, it is worth to remark that the cost function $C(\text{action}, X)$ depends on $W(\tau_1)$ and $W(\tau_3)$ through both its arguments as explained in Table 2, although for economy of notation the adopted symbols do not highlight this dependency.

The expectations at the numerator and denominator of (9) do not allow for closed form expressions. However, they can be computed numerically by using the (10) and (11), respectively.

$$\begin{aligned}
&E\big(C(action, X)\big) \\
&= c_i + c_l + c_p + (c_i + c_d \cdot \tau_3 - c_d \cdot \tau_2) \cdot \mathbb{B}\left(\tfrac{l_1}{l_1+c}; \eta(\tau_1), d\right) \\
&+c_l \cdot \int_0^{l^*} \mathbb{B}\left(\tfrac{L_3-w_1}{L_3+c}; \eta(\tau_3) - \eta(\tau_1); \eta(\tau_1) + d\right) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}w_1 \\
&+\big(c_c - c_p + c_d \cdot \tau_1\big) \cdot \left[1 - \mathbb{B}\left(\tfrac{w_M}{w_M+c}; \eta(\tau_1), d\right)\right] \\
&+ \Big(c_c - c_p + c_l + c_d \cdot (\tau_2 - \tau_1)\Big) \cdot \mathbb{B}\left(\tfrac{L_1}{L_1+c}; \eta(\tau_1), d\right) \\
&-\big(c_c - c_p\big) \cdot \int_{l_1}^{L_1} \mathbb{B}\left(\tfrac{w_M-w_1}{w_M+c}; \eta(\tau_2) - \eta(\tau_1); \eta(\tau_1) + d\right) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}w_1 \\
&+ \Big(c_c - c_p + c_d \cdot (\tau_4 - \tau_3)\Big) \cdot \int_0^{l^*} \mathbb{B}\left(\tfrac{L_3-w_1}{L_3+c}; \eta(\tau_3) - \eta(\tau_1); \eta(\tau_1) + d\right) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}w_1 \\
&- \Big(c_c - c_p + c_d \cdot (\tau_3 - \tau_1)\Big) \cdot \int_0^{l_1} \mathbb{B}\left(\tfrac{w_M-w_1}{w_M+c}; \eta(\tau_3) - \eta(\tau_1); \eta(\tau_1) + d\right) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}w_1 \\
&-\big(c_c - c_p\big) \cdot \int_0^{l^*} \int_{w_1}^{L_3} \mathbb{B}\left(\tfrac{w_M-w_3}{w_M+c}; \eta(\tau_4) - \eta(\tau_3); \eta(\tau_3) + d\right) \cdot f_{W(\tau_3)|W(\tau_1)}(w_3|w_1) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}w_1 \cdot \mathrm{d}w_3 \\
&-c_d \cdot \int_{w_M}^{+\infty} \int_0^{\tau_1} \mathbb{B}\left(\frac{w_M}{w_1}; \eta(x), \eta(\tau_1) - \eta(x)\right) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}x \cdot \mathrm{d}w_1 \\
&-c_d \cdot \int_{l_1}^{L_1} \int_{\tau_1}^{\tau_2} \mathbb{B}\left(\tfrac{w_M-w_1}{w_M+c}; \eta(x) - \eta(\tau_1); \eta(\tau_1) + d\right) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}x \cdot \mathrm{d}w_1 \\
&-c_d \cdot \int_0^{l_1} \int_{w_M}^{+\infty} \int_{\tau_1}^{\tau_3} \mathbb{B}\left(\frac{w_M-w_1}{w_3-w_1}; \eta(x) - \eta(\tau_1), \eta(\tau_3) - \eta(x)\right) \cdot f_{W(\tau_3)|W(\tau_1)}(w_3|w_1) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}x \cdot \mathrm{d}w_1 \cdot \mathrm{d}w_3 \\
&-c_d \cdot \int_0^{l^*} \int_{w_1}^{L_3} \int_{\tau_3}^{\tau_4} \mathbb{B}\left(\tfrac{w_M-w_3}{w_M+c}; \eta(x) - \eta(\tau_3); \eta(\tau_3) + d\right) \cdot f_{W(\tau_3)|W(\tau_1)}(w_3|w_1) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}x \cdot \mathrm{d}w_1 \cdot \mathrm{d}w_3. \quad (10)
\end{aligned}$$

$$\begin{aligned}
&E\{T(action, X)\} = \tau_1 + (\tau_2 - \tau_1) \cdot \mathbb{B}\left(\tfrac{L_1}{L_1+c}; \eta(\tau_1), d\right) + (\tau_3 - \tau_2) \cdot \mathbb{B}\left(\tfrac{l_1}{l_1+c}; \eta(\tau_1), d\right) \\
&+(\tau_4 - \tau_3) \cdot \int_0^{l^*} \mathbb{B}\left(\tfrac{L_3-w_3}{L_3+c}; \eta(\tau_3) - \eta(\tau_1); \eta(\tau_1) + d\right) \cdot f_{W(\tau_1)}(w_1) \cdot \mathrm{d}w_1. \quad (11)
\end{aligned}$$

where $l^* = \min(l_1, L_3)$

Details about the derivation of (10) and (11) are not provided due to space constraints.

4. The example of application

In this section, we apply the proposed new two-inspection maintenance model and compare its performances with those of a comparable special case of the adaptive one-inspection model proposed in (Esposito et al., 2022). Under this latter maintenance policy, a unit undergoes a single inspection at time τ_0 and, based on the measured degradation level, it is decided to either replace the unit immediately or to defer the replacement to a future time,. More in particular, in the latter case, the time at which the replacement is postponed is decided by comparing the measured degradation level to k pre-defined (maintenance) threshold levels, $L_1 < L_2 < \cdots < L_k$. If, at time τ_0, the degradation level is between L_{i-1} and L_i (with $i = 1, 2, \dots, k$, and $L_0 = 0$) then the replacement of the unit is deferred to time τ_i (with $\tau_1 > \tau_2 > \cdots > \tau_k$). If the degradation level at time τ_0 has passed L_k the unit is immediately replaced. The (maintenance) threshold levels $L_1 < L_2 < \cdots < L_k$, the replacement times $\tau_1 > \tau_2 > \cdots > \tau_k$, and the inspection time τ_0 are design parameters whose values are set to minimize the long-run average maintenance cost rate. For this comparison we have set $k = 3$, to have an adaptive one-inspection policy that uses the same number of optional inspection times of the two-inspection policy. Furthermore, we have also compared the performances of the one- and two-inspection policies with those of a (zero-inspection) pure age-based policy.

Even in the case of the one- and zero-inspection policies, failures are supposed to be not self-announcing and replacements are supposed to be allowed only at the predetermined inspection/replacement times.

The parameters of the degradation process are set to the values reported in Table 5, already used also in (Esposito et al., 2022), where a and b should be intended as the parameters of the age function $\eta(t) = a \cdot t^b$ and c and d are the parameters of the pdf (2).

Table 5. Values of the parameters of the degradation process.

a	b	c	d
0.17	1	50	21.25

Hereafter, the two-inspection maintenance policy described in this paper is denoted by $\mathcal{P}_2$ and the corresponding design parameters and long-run average maintenance cost rate will be indicated by ${}_2\boldsymbol{\zeta} = \{{}_2l_1, {}_2L_1, {}_2L_2, {}_2\tau_1, {}_2\tau_2, {}_2\tau_3, {}_2\tau_4\}$ and ${}_2C_\infty({}_2\boldsymbol{\zeta})$, respectively.

In a similar fashion, the adaptive one-inspection policy suggested in (Esposito et al., 2022) is denoted as $\mathcal{P}_1$, and the corresponding design parameters and the long-run average maintenance cost rate will be indicated as ${}_1\boldsymbol{\zeta} = \{{}_1L_1, {}_1L_2, {}_1L_3, {}_1\tau_1, {}_1\tau_2, {}_1\tau_3, {}_1\tau_4\}$ and ${}_1C_\infty({}_1\boldsymbol{\zeta})$. Finally, the zero-inspection (pure age-based) policy is denoted by $\mathcal{P}_0$, and the corresponding parameter and long-run average maintenance cost rate are indicated by ${}_0\boldsymbol{\zeta} = \{{}_0\tau_1\}$ and ${}_0C_\infty({}_0\boldsymbol{\zeta})$, respectively.

The analysis is conducted by varying the values of logistic costs and the inspection costs. In particular, three different setups are adopted:

- Setup $\mathcal{A}$, where both logistic and inspection costs are relatively low (i.e., $c_l = 0.02$, $c_i = 0.05$).
- Setup $\mathcal{B}$, where both logistic and inspection costs are at a moderate level (i.e., $c_l = 0.2$, $c_i = 0.5$).
- Setup $\mathcal{C}$, similar to Setup $\mathcal{A}$ but with a higher inspection cost (i.e., $c_l = 0.4$, $c_i = 1$).

The remaining parameters are held constant throughout the different setups, with $c_p = 2$, $c_c = 6$, and $c_d = 0.2$. The results of the analyses are reported in the following tables.

Table 6. Optimal values of the long-run average maintenance cost rate for the three selected policies and the three selected setups.

	${}_2C_\infty({}_2\boldsymbol{\zeta})$	${}_1C_\infty({}_2\boldsymbol{\zeta})$	${}_0C_\infty({}_2\boldsymbol{\zeta})$
$\mathcal{A}$	0.0382	0.0395	0.0496
$\mathcal{B}$	0.0531	0.0513	0.0516
$\mathcal{C}$	0.0653	0.0636	0.0552

Table 7. Optimal values of the design parameters under the policy $\mathcal{P}_2$ for the three selected setups.

	${}_2l_1$	${}_2L_1$	${}_2L_3$	${}_2\tau_1$	${}_2\tau_2$	${}_2\tau_3$	${}_2\tau_4$
$\mathcal{A}$	16.43	23.00	22.32	39.51	53.47	65.81	86.87
$\mathcal{B}$	2.36	19.71	7.60	48.62	71.12	77.43	82.30
$\mathcal{C}$	1.99	19.98	2.41	52.68	83.57	86.25	128.10

Table 8. Optimal values of the design parameters under the policy $\mathcal{P}_1$ for the three selected setups.

	${}_1L_1$	${}_1L_2$	${}_1L_3$	${}_1\tau_0$	${}_1\tau_1$	${}_1\tau_2$	${}_1\tau_3$
$\mathcal{A}$	11.91	17.40	23.80	42.46	82.42	67.74	55.09
$\mathcal{B}$	12.34	17.44	22.95	46.59	89.80	74.85	62.45
$\mathcal{C}$	12.83	17.65	22.62	50.56	96.64	81.28	68.95

Table 9. Optimal values of the design parameter under the policy $\mathcal{P}_0$ for the three selected setups.

	${}_0\tau_1$
$\mathcal{A}$	54.20
$\mathcal{B}$	54.86
$\mathcal{C}$	56.07

From Table 6, it results that when inspection and logistic costs are low the two-inspection maintenance policy is the one that allows for the lowest long-run average maintenance cost rate, demonstrating the utility of performing the second optional inspection. However, the same table also shows that as inspection and logistic costs increase the advantages of performing inspections decreases. In fact, passing from the setup $\mathcal{A}$ to the setup $\mathcal{C}$, firstly the one-inspection model (under setup $\mathcal{B}$) starts performing better than the two-inspection one and successively the zero-inspection policy (under setup $\mathcal{C}$) start performing better than both the one- and two-inspection policies.

Moreover, it can be observed that for high and moderate values of logistic and inspection costs the policy $\mathcal{P}_2$ tends toward a policy with a single inspection and a singular threshold level. Indeed, the values of ${}_2\tau_1$ and ${}_2l_1$ reported in Table 6, indicate that in the case of setups $\mathcal{B}$ and $\mathcal{C}$ the probability of performing the second inspection approaches 0. Specifically, under the setup $\mathcal{B}$ it is $P(W(48.62) < 2.36) = 1.51 \cdot 10^{-5}$ and under the setup $\mathcal{C}$ it results $P(W(52.68) < 1.99) = 1.04 \cdot 10^{-6}$).

Conclusions

In this paper, a novel maintenance policy has been suggested for degrading units whose degradation paths can be modeled by a gamma process with random effect. Failures are assumed to occur when the degradation level first passes a fixed threshold and are supposed to be not self-announcing.

The proposed maintenance policy involves conducting an initial inspection of the unit at a predetermined time. Hence, based on the measured degradation level, it is decided whether to replace the unit, to defer its replacement, or to perform a second inspection. In the case of a second inspection, based again on the measured degradation level it is decided whether to replace immediately the unit or further defer its replacement. In this latter case the unit will be replaced and no further inspections are conducted. It is assumed that each replacement restores the unit to an "as good as new" state. The performance of the maintenance policy is assessed in terms of the long-run average maintenance cost rate.

The performances of the suggested policy are compared with an ad hoc selected adaptive-one- inspection policy and a standard pure age-based zero-inspection policy.

The comparative study has been developed by considering three different setups that differ in the values of inspection and/or logistic cost only. The performances of the considered policies have been evaluated and compared in terms of long-run average maintenance cost rate. Obtained results demonstrate that depending on the setup either of the considered policies can be preferred to the others. In fact, in particular, it is shown that, as logistic and inspection costs increase, firstly the one-inspection model starts performing better than the two-inspection one and successively the zero-inspection policy start performing better than both the one- and two-inspection policies.

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