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Adaptive Maintenance Policy For Gamma Degrading Units With Bathtub-Shaped Degradation Rate Function In The Presence Of Random Effect

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Abstract

The main goal of this paper is to investigate, in the field of maintenance optimization, the performances of a new gamma process with random effect and bathtub-shaped degradation rate function. Maintenance costs are determined by applying an adaptive hybrid age-/condition-based policy recently proposed in the literature, which consists in measuring the degradation level of the unit at a first (age-based) inspection time and using a condition-based rule to decide whether to immediately replace the unit or to postpone its replacement to a future time. The policy is denominated adaptive, since the future replacement time is decided, unit by unit, based on the outcome of the inspection. The optimal maintenance policy is defined by minimizing the long-run average maintenance cost rate. After each replacement the unit is considered as good as new. The lifetime of the unit is defined by using a failure threshold model. Maintenance costs are computed by accounting for preventive replacement cost, corrective replacement cost, inspection cost, logistic cost, and downtime cost (which depends on the time spent in a failed state). An example of application demonstrates the affordability of the approach. A study is also performed, by using simulated data generated under the model with bathtub-shaped degradation rate function, to investigate the effect on maintenance costs of fitting the data by using a gamma process with random effect where the age function has a classical power law shape.

Keywords: gamma process, random effect, bathtub-shaped degradation rate function, age based and condition based maintenance, average long run maintenance cost rate

1. Introduction

In the literature, it is often observed that the degradation rate (here intended as the derivative of the mean function) of real-world technological units shows three phases: a first (early) phase where the degradation rate decreases, a second one where it is constant, and a final third (catastrophic or degenerative) phase where it increases; e.g., see (Gertsbakh and Kordonskiy, 1969). Nonetheless, the vast majority of degradation models proposed in the literature are not able to describe this kind of behavior, a circumstance that could limit their effectiveness in the case of degrading units whose operational life shows all the mentioned three phases.

To fill this gap, (Giorgio et al., 2023) and (Piscopo et al., 2023) have recently proposed degradation processes (a Wiener and a gamma process, respectively) that can be used to describe degradation phenomena characterized by a bathtub-shaped degradation rate function.

In (Giorgio et al., 2023) and (Piscopo et al., 2023) the utility and affordability of these models is shown by applying them to the MOSFETs data presented in (Lu et al., 1997). In these papers, the performances of the proposed models were evaluated in terms of their ability to fit the available data and predict the remaining useful life of the considered degrading units. Here, we propose a new gamma process with bathtub-shaped degradation rate and random effect inspired by the one suggested in (Piscopo et al., 2023) and apply it to maintenance optimization.

Specifically, here we adopt a customized version of the gamma process with random effect firstly proposed by (Lawless and Crowder, 2004), where, given the values of the random shape parameter, the gamma processes have a bathtub-shaped degradation rate. Then, we compute maintenance costs by using the maintenance policy proposed in (Esposito et al., 2022), which consists in performing a single inspection at a predetermined time and, based on its outcome, immediately replacing the unit or postponing its replacement to a future time. In case of postponement, the replacement time will be determined adaptively based on the measured degradation level. At this future replacement time, the unit will be replaced regardless of its degradation level, without any additional inspections. According to the failure threshold model, it is assumed that a unit fails when its degradation level passes an assigned failure threshold.

When the degradation rate function is bathtub-shaped, the mean function has an inverse S-shaped behavior, which implies the presence of an inflection point where the degradation rate changes from decreasing to increasing. When the scale parameter of the gamma process is treated as a random variable (as we have assumed in this paper), degradation processes of different units evolve at different speeds. In particular, weak units (i.e., units whose degradation progresses relatively fast) tend to fail earlier with respect to strong ones (i.e., units whose degradation progresses relatively slow). Thus, understanding if a unit is weak or strong could be very useful information making effective maintenance decisions.

In fact, the key idea behind the proposed maintenance policy is to exploit the inspection to timely assess if a unit is weak or strong and hence to define for it an ad hoc (unit-specific) replacement time which accounts for the rapidity with which its degradation evolves over time. Obviously, the objective is to timely carry out the replacement of the weak units and to postpone at later times the replacement of the stronger ones.

The affordability of the approach is demonstrated by developing a realistic applicative example. Moreover, a short Monte Carlo study is also performed, by using simulated data generated under the model with bathtub-shaped degradation rate function, to investigate the effect on maintenance costs of misspecifying the true model with a gamma process with random effect and power law age function.

The rest of the paper is structured as follows. Section 2 illustrates the adopted gamma process with bathtub-shaped degradation rate and random effect. Section 3 describes the adopted maintenance policy. Section 4 deals with the formulation of the long-run average maintenance cost rate. Section 5 presents the results of an example of application of the proposed approach. Section 6 gives some concluding remarks.

2. The degradation process

In this paper, the degradation process $\{W(t), t \geq 0\}$ is a non-homogeneous gamma process with age function:

$$\eta(t) = a_1 \cdot t^{b_1} + a_2 \cdot t^{b_2}, \tag{1}$$

where the scale parameter λ is assumed to vary from unit to unit according to a gamma random variable with scale parameter c and shape parameter d.
To remark that the scale parameter is random, hereinafter we will indicate it by Λ and its realization by λ.

Under this setting, the probability density function (pdf) of the generic degradation increment $\Delta W(t_1, t_2) = W(t_2) - W(t_1)$, given the value λ of the random scale parameter Λ, and the pdf of Λ can be respectively expressed as:

$$f_{\Delta W(t_1,t_2)|\Lambda}(\delta|\lambda) = \frac{\lambda^{\Delta\eta(t_1,t_2)} \cdot \delta^{\Delta\eta(t_1,t_2)-1}}{\Gamma(\Delta\eta(t_1,t_2))} \cdot e^{-\lambda\cdot\delta}, \delta \geq 0, \tag{2}$$

and:

$$f_\Lambda(\lambda) = \frac{c^d \cdot \lambda^{d-1}}{\Gamma(d)} \cdot e^{-c\cdot\lambda}, \quad \lambda, c, d > 0. \tag{3}$$

where $\Gamma(\cdot)$ denotes the complete gamma function, and $\Delta\eta(t_1, t_2) = \eta(t_2) - \eta(t_1)$

The resulting model is Markovian, hence (1)-(3), together with an initial condition (here $W(0) = 0$), fully define the model.

Under this setting, the marginal pdf of $W(t)$ can be expressed as:

$$f_{W(t)}(w) = \frac{c^d \cdot w^{\eta(t)-1}}{\mathrm{B}(\eta(t),d)\cdot(w+c)^{\eta(t)+d}}, \quad w \geq 0,$$

and the corresponding marginal cumulative distribution function (cdf) results in:

$$F_{W(t)}(w) = \mathbb{B}\left(\frac{w}{w+c}; \eta(t), d\right), \quad w \geq 0,$$

where:

$$\mathbb{B}(z;\alpha,\beta)=\int_0^z x^{\alpha-1}\cdot(1-x)^{\beta-1}\cdot\frac{dx}{B(\alpha,\beta)},\ \ \alpha,\beta>0,$$

is the regularized beta function and $\mathrm{B}(r,s)=\Gamma(r)\cdot\Gamma(s)/\Gamma(r+s)$ is the beta function. In addition, the conditional pdf of the increment $\Delta W(t_1,\ t_2)$, given $W(t_1)=w_1$, can be formulated as:

$$f_{\Delta W(t_1,t_2)|W(t_1)}(\delta|w_1)=\frac{1}{B(\Delta\eta(t_1,t_2),\eta(t_1)+d)}\cdot\frac{(w_1+c)^{\eta(t_1)+d}\cdot\delta^{\Delta\eta(t_1,t_2)-1}}{(w_1+\delta+c)^{\eta(t_2)+d}}\ ,\quad \delta\geq 0 \tag{4}$$

while the corresponding (conditional) cdf can be expressed as:

$$F_{\Delta W(t_1,t_2)|W(t_1)}(\delta|w_1)=\mathbb{B}\left(\frac{\delta}{\delta+w_1+c};\Delta\eta(t_1,t_2),\eta(t_1)+d\right),\quad \delta\geq 0. \tag{5}$$

As already mentioned above, we assume that a unit fails when its degradation level passes an assigned failure threshold, hereinafter indicated by w_M. Thus, given that the hidden process $\{W(t),t\geq 0\}$ is monotone increasing, the useful life X of the unit (i.e., its time to failure):

$$X=\inf\{x\colon W(x)>w_M\}$$

can be defined as the first and sole passage time of $\{W(t),t\geq 0\}$ to the failure threshold w_M.

In the rest of this section, we provide some results that involve the lifetime X which have been used to formulate the expression of the long-run average maintenance cost rate reported in Section 4.

From the conditional cdf (5), it is possible to readily formulate the following conditional cdf of X given $W(\tau)=w_\tau$ in the cases where $w_\tau\leq w_M$:

$$F_{X|W(\tau)}(x|w_\tau)=P(X\leq x|W(\tau)=w_\tau)$$

$$=P(W(x)>w_M|W(\tau)=w_\tau)=\begin{cases}0, & w_\tau\leq w_\mathrm{M},\ \ x\leq\tau\\ 1-\mathbb{B}\left(\frac{w_M-w_\tau}{w_M+c};\ \Delta\eta(\tau,x),\eta(\tau)+d\right), & w_\tau\leq w_\mathrm{M},\ \ x>\tau\end{cases} \tag{6}$$

where τ should be intended as a generic assigned reference time. Indeed, the first equality can be explained by observing that, since the process $\{W(t),t\geq 0\}$ is monotone increasing, the event $\{X\leq x\}$ is equivalent to the event $\{W(x)>w_\mathrm{M}\}$.

Differently, in the case where $w_\tau>w_\mathrm{M}$ the conditional cdf of X given $W(\tau)=w_\tau$ can be expressed as in (7):

$$F_{X|W(\tau)}(x|w_\tau)=P(X\leq x|W(\tau)=w_\tau)$$

$$=P(W(x)>w_M|W(\tau)=w_\tau)=\begin{cases}1-\mathbb{B}\left(\frac{w_M}{w_\tau};\ \eta(x),\Delta\eta(x,\tau)\right), & w_\tau>w_M,\quad x\leq\tau\\ 1, & w_\tau>w_M,\ \ x>\tau\end{cases}. \tag{7}$$

Note that, for the sake of economy of notation, the cdfs in (6) and (7) have been indicated by using the same symbol. However, as it is explicitly specified in the text, the (6) should be used if and only if $w_\tau\leq w_M$ and (7) if and only if $w_\tau>w_M$.

From (7), when $w_\tau>w_M$, the conditional mean of X, given $W(\tau)=w_\tau$, can be expressed as:

$$E\{X|W(\tau)=w_\tau\}=\int_0^\tau x\cdot f_{X|W(\tau)}(x|w_\tau)\cdot\mathrm{d}x=\int_0^\tau\mathbb{B}\left(\frac{w_M}{w_\tau};\ \eta(x),\Delta\eta(x,\tau)\right)\cdot\mathrm{d}x.$$

Finally, from (6), when $w_\tau\leq w_M$, given $W(\tau)=w_\tau$, the conditional mean of the variable $g(X)$ defined by the following transformation:

$$g(X)=\begin{cases}\tau+\Delta\tau-X, & X\leq\tau+\Delta\tau\\ 0, & X>\tau+\Delta\tau\end{cases}$$

can be computed as:

$$E\{g(X)|W(\tau)=w_\tau\}$$

$$=\int_\tau^{\tau+\Delta\tau}(\tau+\Delta\tau-x)\cdot f_{X|W(\tau)}(x|w_\tau)\cdot\mathrm{d}x=\Delta\tau-\int_\tau^{\tau+\Delta\tau}\mathbb{B}\left(\frac{w_M-w_\tau}{w_M+c};\ \eta(x)-\eta(\tau),\eta(\tau)+d\right)\cdot\mathrm{d}x.$$

3. The maintenance policy and the cost model

In this paper, we consider a single unit whose degradation evolution can be described by the process presented in Section 2. The policy exploits the degradation information gathered by means of a single inspection performed at a predetermined time, which returns an exact measurement of the degradation level of the unit. This piece of information is then exploited to decide whether to immediately replace the unit or to postpone its replacement according to a condition-based rule. Moreover, it is assumed that:

- inspections are instantaneous and non-destructive;
- failure is not self-announcing. Hence, failed units continue to operate, albeit with reduced performances and/or additional costs;
- both corrective and preventive replacements restore the unit to an "as good as new" state. Hence, replacements are renewal points of a renewal process and the time between two successive replacements can be intended as the length of a cycle (i.e., the maintenance cycle).

Table 1 summarizes the condition-based rule, where $L_1 < L_2 < \cdots < L_k \leq w_M$.

Table 1. The condition-based rule.

Measurement at τ	Decision
$w_\tau > L_k$	Immediate replacement
$L_{k-1} < w_\tau \leq L_k$	Postpone replacement to $\tau + \Delta\tau_k$
$\vdots$	$\vdots$
$L_1 < w_\tau \leq L_2$	Postpone replacement to $\tau + \Delta\tau_2$
$w_\tau \leq L_1$	Postpone replacement to $\tau + \Delta\tau_1$

All the variables τ, $L_1 < L_2 < \cdots < L_k$ and $\Delta\tau_1 > \Delta\tau_2 > \cdots > \Delta\tau_k$ should be intended as design parameters. The set of design parameters is denoted by $\boldsymbol{\xi} = \{\tau,\ L_1, \ldots L_k, \Delta\tau_1, \ldots, \Delta\tau_k\}$, and the optimal set $\boldsymbol{\xi}^* = \{\tau^*, L_1^*, \ldots L_k^*, \Delta\tau_1^*, \ldots, \Delta\tau_k^*\}$ should be determined by the policy based on economic considerations.

As the parameter k is increased, the policy's effectiveness improves at the cost of escalating computational demands. This paper addresses the determination of the optimal value for k by iteratively employing an optimization procedure with progressively larger k values. The objective is to strike a balance between simplicity and efficacy. The optimal maintenance policy is characterized as the one that, for the chosen k value, minimizes the long-run average maintenance cost rate computed using the renewal/reward theorem.

All the possible scenarios, together with the corresponding maintenance actions, maintenance costs, and cycle lengths, are listed in Table 2, where $w_{\tau+\Delta\tau_j}$ denotes the state at $\tau + \Delta\tau_j$, X denotes the unit lifetime, and c_l, c_i, c_p, c_c, and c_d denote the logistic, inspection, preventive replacement, corrective replacement, and downtime cost rate, respectively.

The logistic cost is supposed to be sustained each time a maintenance action is carried out, whereas the downtime cost is computed as the product of the fixed downtime cost rate c_d and the time spent in a failed state (the time elapsing between the failure of the units and its corrective replacement, i.e., the downtime).

It is worth mentioning that, despite the adopted notation not highlighting it, $T(w_\tau)$ and $C(w_\tau, X)$ functionally depend on the vector of design parameters $\boldsymbol{\xi}$. Moreover, coherently with the assumption of not self-announcing failures, the lifetime X is always denoted with the capital letter to indicate that, even when w_τ is known, X should still be intended as a random variable.

Table 2. Possible scenarios and corresponding maintenance costs.

Scenario	Maintenance action	Cycle length $T(w_\tau)$	Maintenance cost $C(w_\tau, X)$
$L_k < w_\tau \leq w_M$	Preventive replacement at τ	τ	$c_l + c_i + c_p$
$w_\tau > w_M$	Corrective replacement at τ	τ	$c_l + c_i + c_c + c_d \cdot (\tau - X)$
$L_{k-1} < w_\tau \leq L_k$ and $w_{\tau+\Delta\tau_k} \leq w_M$	Preventive replacement at $\tau + \Delta\tau_k$	$\tau + \Delta\tau_k$	$2 \cdot c_l + c_i + c_p$
$L_{k-1} < w_\tau \leq L_k$ and $w_{\tau+\Delta\tau_k} > w_M$	Corrective replacement at $\tau + \Delta\tau_k$	$\tau + \Delta\tau_k$	$2 \cdot c_l + c_i + c_c + c_d \cdot (\tau + \Delta\tau_k - X)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$w_\tau \leq L_1$ and $w_{\tau+\Delta\tau_1} \leq w_M$	Preventive replacement at $\tau + \Delta\tau_1$	$\tau + \Delta\tau_1$	$2 \cdot c_l + c_i + c_p$
$w_\tau \leq L_1$ and $w_{\tau+\Delta\tau_1} > w_M$	Corrective replacement at $\tau + \Delta\tau_1$	$\tau + \Delta\tau_1$	$2 \cdot c_l + c_i + c_c + c_d \cdot (\tau + \Delta\tau_1 - X)$

4. Formulation of the long-run average maintenance cost rate

The long-run average maintenance cost rate is computed by using the renewal/reward theorem (e.g., see (Ross, 1983)) as:

$$C_\infty(\boldsymbol{\xi}) = \frac{E\{C(W(\tau), X)\}}{E\{T(W(\tau))\}}, \tag{8}$$

where expectations have to be taken with respect to all the variables that are within the curly braces. The expected values included in (8) are not available in closed form but can be efficiently computed via (9) and (10). Specifically, $E\{T(W(\tau))\}$ can be computed as:

$$E\{T(W(\tau))\} = \int_0^\infty T(w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}w_\tau$$

$$= \sum_{j=1}^{k} \int_{L_{j-1}}^{L_j} T(w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}w_\tau + \int_{L_k}^{\infty} T(w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}w_\tau$$

$$= \sum_{j=1}^{k} \int_{L_{j-1}}^{L_j} (\tau + \Delta\tau_j) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}w_\tau + \int_{L_k}^{\infty} \tau \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}w_\tau$$

$$= \tau + \sum_{j=1}^{k} \Delta\tau_j \cdot \left[F_{W(\tau)}(L_j) - F_{W(\tau)}(L_{j-1})\right]. \tag{9}$$

Similarly, $E\{C(W(\tau), X)\}$ can be computed as:

$$\{C(W(\tau), X)\} = \int_0^\infty \int_0^\infty C(w_\tau, x) \cdot f_{X|W(\tau)}(x|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}x \cdot \mathrm{d}w_\tau$$

$$= \sum_{j=1}^{k} \int_{L_{j-1}}^{L_j} \int_{\tau}^{\tau+\Delta\tau_j} C(w_\tau, x) \cdot f_{X|W(\tau)}(x|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}x \cdot \mathrm{d}w_\tau$$

$$+ \sum_{j=1}^{k} \int_{L_{j-1}}^{L_j} \int_{\tau+\Delta\tau_j}^{\infty} C(w_\tau, x) \cdot f_{X|W(\tau)}(x|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}x \cdot \mathrm{d}w_\tau$$

$$+ \int_{L_k}^{w_M} \int_{\tau}^{\infty} C(w_\tau, x) \cdot f_{X|W(\tau)}(x|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}x \cdot \mathrm{d}w_\tau$$

$$+ \int_{w_M}^{\infty} \int_0^{\tau} C(w_\tau, x) \cdot f_{X|W(\tau)}(x|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}x \cdot \mathrm{d}w_\tau,$$

which, from Table 2 (and a few cumbersome but simple manipulations) becomes:

$$E\{C(W(\tau), X)\} = c_l + c_i + c_p + c_l \cdot F_{W(\tau)}(L_k) + (c_c - c_p + c_d \cdot \tau) \cdot \left[1 - F_{W(\tau)}(w_M)\right]$$

$$+ (c_c - c_p) \cdot \sum_{j=1}^{k} \int_{L_{j-1}}^{L_j} [1 - F_{\Delta W(\tau,\tau+\Delta\tau)|W(\tau)}(w_M - w_\tau|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}w_\tau$$

$$- c_d \cdot \sum_{j=1}^{k} \int_{L_{j-1}}^{L_j} \int_{\tau}^{\tau+\Delta\tau_j} F_{\Delta W(\tau,x)|W(\tau)}(w_M - w_\tau|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}x \cdot \mathrm{d}w_\tau$$

$$+ c_d \cdot \sum_{j=1}^{k} \Delta\tau_j \cdot \left[F_{W(\tau)}(L_j) - F_{W(\tau)}(L_{j-1})\right] - c_d \cdot \int_{w_M}^{\infty} \int_0^{\tau} F_{W(x)|W(\tau)}(w_M|w_\tau) \cdot f_{W(\tau)}(w_\tau) \cdot \mathrm{d}x \cdot \mathrm{d}w_\tau. \tag{10}$$

5. Example of application and comparative analysis

5.1. Example of application

In order to show the affordability of the proposed approach, in this section we present an example of application, developed by considering a realistic experimental scenario. The parameters of degradation and cost models are set to the values reported in Tables 3 and 4, respectively. Units are assumed to fail when their degradation level passes the threshold $w_M = 30$.

Table 3. Parameters of the degradation model.

a_1	b_1	a_2	b_2	c	d
0.8	0.5	0.02	3	50	21.25

Table 4. Parameters of the cost model.

c_l	c_i	c_p	c_c	c_d
0.2	0.5	1	6	0.2

As already remarked above, the parameter k (i.e., the number of classes) of the adaptive maintenance model can be intended as a “hyperparameter” which, when increased, improves the performances of the policy at the cost of a heavier computational burden. Figure 1 highlights this situation. Indeed, it reports (in blue, solid line, left vertical axis) the optimal long-run average maintenance cost rate as a function of k. The same figure also reports (in red, dashed line, right vertical axis), as a measure of the computational burden, the time (in seconds) needed to find the optimum on our machine of reference.

Figure 1 shows that, as expected, the optimal cost decreases with k. However, it plateaus after $k = 10$, while the computational burden keeps increasing. For this reason, all subsequent analyses will be performed with k set to 10.

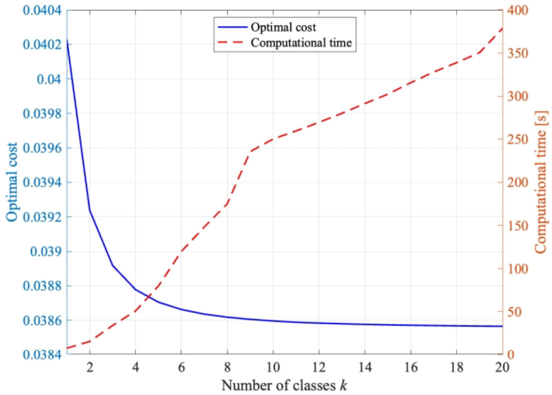

Fig. 1. Optimal long-run average maintenance cost rate (blue solid line, left axis) and computational time (red dashed line, right axis) as a function of the number of classes k.

Table 5 reports the optimal values of the design parameters that minimize the long-run average maintenance cost rate in the case of $k = 10$ classes. The corresponding minimum long-run average maintenance cost rate is 0.0386. These values are used to develop the comparative analyses reported in the next subsection.

Table 5. Values of the optimal design parameters and corresponding optimal long-run average maintenance cost rate.

τ	$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$	$\Delta\tau_4$	$\Delta\tau_5$	$\Delta\tau_6$	$\Delta\tau_7$	$\Delta\tau_8$	$\Delta\tau_9$	$\Delta\tau_{10}$
36.42	45.34	40.73	36.85	33.36	29.83	26.45	22.95	19.29	15.50	11.69
C_∞^*	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
0.0386	4.98	6.86	8.55	10.05	11.57	13.07	14.64	16.29	18.01	19.77

Figure 2 shows, via a step function, the optimal value of the replacement time $\Delta\tau^*$ as a function of the degradation level measured at τ, denoted by w_τ in the case of $k = 10$ classes. This illustrates how the policy adaptively assigns high values of $\Delta\tau^*$ to units which at τ are barely degraded, and progressively smaller values as w_τ increases.

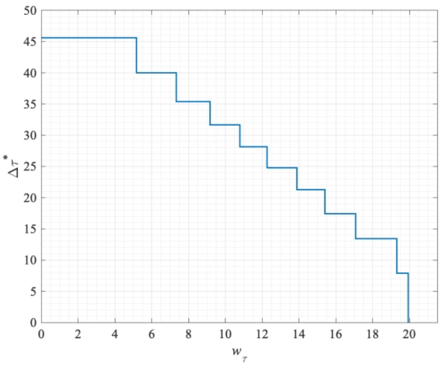

Fig. 2. Optimal replacement time as a function of the measured degradation level at τ in the case of $k = 10$ classes.

5.2. Comparative analysis

In this section, we aim to assess the impact on the long-run average maintenance cost rate of neglecting the circumstance that the degradation rate is bathtub-shaped. For the comparison, we suppose that the proposed gamma process with random effect and bathtub-shaped degradation rate (which is assumed to be the true model) is misspecified with a gamma process with random effect where the age function has the classical power-law expression $\eta(t) = a \cdot t^b$.

Hereinafter, we will denote the model with bathtub-shaped degradation rate by M1 and the alternative model by M2. It is worth mentioning that M2 can be obtained as a special case of M1 when a_2 is set to 0. Under both the models the scale parameter is assumed to be gamma distributed with pdf (3).

To perform the analysis, we have generated 100 synthetic datasets under the model M1, with model parameters set to the values reported in Table 1. Each data set consists of 200 degradation measurements, obtained by observing the degradation level of 8 units at 25 different equally spaced inspection times, $t_{i,1} = 4, t_{i,2} = 8, \dots, t_{i,25} = 100\ \forall i = 1, \dots, 8$.

Hence, we have fitted both the models M1 and M2 to each synthetic dataset by means of the Maximum Likelihood (ML) method, obtaining 200 "estimated" models, 100 estimates of the model M1 and 100 of the model M2. Subsequently, we have optimized the considered maintenance policy by using all the estimated models, obtaining 200 estimates of the optimal set of design parameters $\boldsymbol{\xi}^* = \{\tau^*, L_1^*, \dots L_k^*, \Delta\tau_1^*, \dots, \Delta\tau_k^*\}$.

The likelihood is formulated as in (11):

$$L(\boldsymbol{\vartheta}; \boldsymbol{w}) = \prod_{i=1}^{8} \prod_{j=1}^{25} f_{\Delta W(t_{j-1}, t_j)|W(t_j)}\left(w_{i,j} - w_{i,j-1} | w_{i,j}\right) \tag{11}$$

where $f_{\Delta W(t_{j-1}, t_j)|W(t_j)}\left(w_{i,j} - w_{i,j-1} | w_{i,j}\right)$ is the pdf in (4), $W\left(t_{i,j}\right)$ is the degradation level of the unit i at the measurement epoch $t_{i,j}$, $w_{i,j}$ is its realization, $\Delta W\left(t_{i,j-1}, t_{i,j}\right) = W\left(t_{i,j}\right) - W\left(t_{i,j-1}\right)$, $\boldsymbol{w} = \left\{w_{1,1}, \dots, w_{1,25}, \dots, w_{8,1}, \dots, w_{8,25}\right\}$, $w_{i,0} = 0$, $t_{i,0} = 0$, and $\eta(t)$ is either the function in (1) or the power law function $a \cdot t^b$, depending on the model used to fit the data (i.e., either M1 or M2, respectively).

The vector of model parameters is denoted by $\boldsymbol{\vartheta}$ and corresponds to $\boldsymbol{\vartheta} = \{a_1, b_1, a_2, b_2, c, d\}$ under the model M1 and to $\boldsymbol{\vartheta} = \{a, b, c, d\}$ under model M2. The ML estimate $\widehat{\boldsymbol{\vartheta}}$ is the value of $\boldsymbol{\vartheta}$ which maximizes (11), given $\boldsymbol{w}$.

Finally, under each estimated model we have determined the value of the set of design parameters which minimizes maintenance costs computed as in (8). Obtained results have been used to compute the following indices:

$$MRD_M^{(1)} = \frac{\sum_{h=1}^{100} \frac{C_\infty\left(\hat{\xi}_{M,h}^*\right) - C_\infty(\xi^*)}{C_\infty(\xi^*)}}{100} \tag{12}$$

$$SDRD_M^{(1)} = \sqrt{\frac{\sum_{h=1}^{100} \left[\frac{C_\infty\left(\hat{\xi}_{M,h}^*\right) - C_\infty(\xi^*)}{C_\infty(\xi^*)} - MRD_M^{(1)}\right]^2}{100}}, \tag{13}$$

$$MRD_M^{(2)} = \frac{\sum_{h=1}^{100} \frac{\hat{C}_{\infty,M,h}\left(\hat{\xi}_{M,h}^*\right) - C_\infty(\hat{\xi}_{M,h}^*)}{C_\infty(\hat{\xi}_{M,h}^*)}}{100} \tag{14}$$

$$SDRD_M^{(2)} = \sqrt{\frac{\sum_{h=1}^{100} \left[\frac{\hat{C}_{\infty,M,h}\left(\hat{\xi}_{M,h}^*\right) - C_\infty(\hat{\xi}_{M,h}^*)}{C_\infty(\hat{\xi}_{M,h}^*)} - MRD_M^{(2)}\right]^2}{100}}, \tag{15}$$

where:

- $C_\infty(\cdot)$ is the true value of the long-run average maintenance cost rate (i.e., the one computed by using the true process);
- $\boldsymbol{\xi}^*$ is the true optimal value of the vector of design parameters $\boldsymbol{\xi}$ (i.e., the value of $\boldsymbol{\xi}$ that minimizes $C_\infty(\boldsymbol{\xi})$);
- $C_\infty(\boldsymbol{\xi}^*)$ is the true minimum long-run average maintenance cost rate;
- $\hat{C}_{\infty,M,h}(\cdot)$ is the long-run average maintenance cost rate computed by using the model M with model parameters set at their MLEs obtained (under the model M) from the hth dataset (the hat "$\widehat{\ }$" indicates that $\hat{C}_{\infty,M,h}(\cdot)$ can be intended as an estimate of $C_\infty(\cdot)$);
- $\hat{\boldsymbol{\xi}}_{M,h}^*$ is the value of $\boldsymbol{\xi}$ that minimizes $\hat{C}_{\infty,M,h}(\boldsymbol{\xi})$ (the hat "$\widehat{\ }$" indicates that $\hat{\boldsymbol{\xi}}_{M,h}^*$ can be intended as an estimate of $\boldsymbol{\xi}^*$);

- $\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h})$ is the (estimated) minimum long-run average maintenance cost rate computed by using the model M with model parameters set at their MLEs obtained (under the model M) from the hth dataset;
- $C_\infty(\hat{\xi}^*_{M,h})$ is the true long-run average maintenance cost rate obtained by setting $\xi = \hat{\xi}^*_{M,h}$ (this cost is evaluated by using the true process).

The cost $C_\infty(\hat{\xi}^*_{M,h})$ differs from $C_\infty(\xi^*)$ because $\hat{\xi}^*_{M,h}$ is obtained by minimizing $\hat{C}_{\infty,M,h}(\cdot)$ instead of $C_\infty(\cdot)$. $\hat{C}_{\infty,M,h}(\cdot)$ differs from $C_\infty(\cdot)$ because model parameters are estimated. In addition, when $\mathrm{M} = \mathrm{M2}$, the model used to compute maintenance costs is not the right one.
In other words, if we suppose that $C_\infty(\cdot)$ is the "true" cost function, then $C_\infty(\xi^*)$ is the true minimum, $C_\infty(\hat{\xi}^*_{M,\mathrm{h}})$ is the long-run average maintenance cost rate that is actually sustained when the design parameters of the policy are set to $\hat{\xi}^*_{M,h}$, while $\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h})$ is its estimate.

The index $MRD_M^{(1)}$ in (12) provides the mean of the relative difference between the long-run average maintenance cost rate $C_\infty(\hat{\xi}^*_{M,h})$ computed under the true model by setting the design parameter at the estimated optimal value $\hat{\xi}^*_{M,h}$ (determined under the estimated model M), and the true minimum long-run average maintenance cost rate $C_\infty(\xi^*)$, computed under the true model by setting the design parameters at the true optimal value ξ^*. The index $SDRD_M^{(1)}$ in (13) is the (empirical) standard deviation of the relative difference $\left(C_\infty(\hat{\xi}^*_{M,h}) - C_\infty(\xi^*)\right)/C_\infty(\xi^*)$ (i.e., $SDRD_M^{(1)}$ indicates how, as the dataset varies, $\left(C_\infty(\hat{\xi}^*_{M,h}) - C_\infty(\xi^*)\right)/C_\infty(\xi^*)$ deviates from its mean).

The index $MRD_M^{(2)}$ in (14) provides the mean of the relative difference between the long-run average maintenance cost rate $\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h})$ computed under the estimated model M by setting the design parameter at the estimated optimal value $\hat{\xi}^*_{M,h}$ (determined under the estimated model M), and the true long-run average maintenance cost $C_\infty(\xi^*)$, computed under the true model by setting design parameters at their estimated optimal value $\hat{\xi}^*_{M,h}$, determined under the estimated model M. The index $SDRD_M^{(1)}$ in (15) is the (empirical) standard deviation of the relative difference $(\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h}) - C_\infty(\hat{\xi}^*_{M,h}))/C_\infty(\hat{\xi}^*_{M,h})$.

Obtained results are reported in Table 6.

Table 6. Results of the study

	$MRD_M^{(1)}$	$SDRD_M^{(1)}$	$MRD_M^{(2)}$	$SDRD_M^{(2)}$
M1	0.0361	0.0454	−0.0496	0.1656
M2	0.0513	0.0685	0.0903	0.1683

The value of 0.0361 of $MRD_M^{(1)}$ in the first row of the table shows that adopting the estimated model M1 in place of the true model leads to a cost that, in mean, is 3.61% higher than the true optimum. On the other hand, the value of $MRD_M^{(1)}$in the second row shows that adopting the estimated model M2 leads to a cost increase that, in mean, is 5.13% higher than the true optimum, with a cost increase that is almost twice the cost increase caused by using the estimated model M1. The values of $SDRD_M^{(1)}$ reported in the second column of Table 4 shows that, under the considered setting, these percent increases vary sensibly from dataset to dataset (i.e., the ratio between the standard deviation of the relative difference $\left(C_\infty(\hat{\xi}^*_{M,h}) - C_\infty(\xi^*)\right)/C_\infty(\xi^*)$ and its mean, under both models, is about 1.3).

Differently, the negative value of $MRD_M^{(2)}$ reported in the first row of Table 6 indicates that the long-run average maintenance cost rate $\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h})$ computed under the estimated model M1, underestimates (in mean) the true long-run average maintenance cost rate by 4.96%. Similarly, the positive value of $MRD_M^{(2)}$ reported in the second row of Table 6 indicates that the long-run average maintenance cost rate $\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h})$ computed under the estimated model M2, overestimates (in mean) the true long-run average maintenance cost rate by 9.03%. The values of $SDRD_M^{(2)}$ reported in the last column indicates that, under the considered setting, the percent difference $(\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h}) - C_\infty(\hat{\xi}^*_{M,h}))/C_\infty(\hat{\xi}^*_{M,h})$ both under the model M1 and M2 vary from dataset to dataset a bit more than the percent difference $\left(C_\infty(\hat{\xi}^*_{M,h}) - C_\infty(\xi^*)\right)/C_\infty(\xi^*)$ (i.e., the ratio between the standard deviation of the relative difference $(\hat{C}_{\infty,M,h}(\hat{\xi}^*_{M,h}) - C_\infty(\hat{\xi}^*_{M,h}))/C_\infty(\hat{\xi}^*_{M,h})$ and its mean is about is equal to 3.34 in the case of the model M1 and to 1.86 in the case of the model M2).

Conclusions

In this paper, we have investigated the performances of a new gamma process with random effect and bathtub-shaped degradation rate function in maintenance optimization. We adopted a maintenance policy that consists in performing, at a predetermined (age-based) time, a single inspection aimed at measuring the degradation level of the unit and in using a condition-based rule to decide whether to immediately replace the unit or to postpone its replacement to a future time. The future replacement time is decided, unit by unit, based on the outcome of the inspection. The optimal maintenance policy is defined by minimizing the long-run average maintenance cost rate. A unit is considered failed when its degradation level passes an assigned failure threshold. Each replacement is assumed to restore the unit to an as good as new condition. Maintenance costs are computed by accounting for preventive replacement cost, corrective replacement cost, inspection cost, logistic cost, and downtime cost (which depends on the time spent in a failed state). The affordability of the approach is demonstrated via a realistic example of application. A Monte Carlo study is also performed, by using simulated data generated under the model with a bathtub-shaped degradation rate function, to investigate the impact on maintenance costs of fitting the data by using a gamma process with random effect where the age function has a classical power law shape. Obtained results show that neglecting the circumstance that the degradation rate is bathtub-shaped leads to, on average, higher maintenance costs. Moreover, it also leads to overestimating the true cost.

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