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Circular And Toroidal Connected (r,2)-Out-Of-(m,n):F Systems: Exact And Asymptotic Results

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Abstract

Connected (r,s)-out-of-(m,n):F systems consist in m x n elements arranged in n rows and m columns. They fail if *all* elements in a block r x s fail. Such systems can successfully describe real-life equipment, and are useful to facilitate X-ray and disease diagnostics, and assess the reliability of electronic devices, as well as the security of communications and property, etc. Computing their exact availability in the general case can however be a rather difficult task, especially in very large systems (m and n large). The majority of published work consider connected systems as two-dimensional grids. Recently, exact solutions have been proposed for various (r,s) values with m reasonably large, while n can be arbitrarily large. Based on these results, an analytical, asymptotic expansion has been given for large m and n, which was shown to be in excellent agreement for m as low as 4. In this paper, we turn to the configurations of circular and toroidal (r,s)-out-of-(m,n):F systems, and show how their reliability or availability may differ from the usual "rectangular" case. Exact, analytical recurrence relations are found again, as well as slightly different asymptotic expression.

Keywords: cellular network, connected (r,s)-out-of-(m,n):F lattice system, network reliability, availability, generating function, asymptotic expansion

1. Introduction

A generalization of the well-known k-out-of-n systems was initially proposed by (Salvia and Lasher, 1990; Ksir, 1992; Boehme et al., 1992; Zuo, 1993; Preuss and Boehme, 1994) for two-dimensional systems. A (r,s)-out-of-(m,n):F system consists in m x n elements arranged in n rows and m columns; the system fails if all elements in a block r x s fail. A simple description of a telecommunication network may be performed by considering such a system.

Several other practical applications were recognized early on by (Salvia and Lasher, 1990), among which the reliability of electronic devices as a straightforward application of their model. This has remained true in the following decades (Chang and Mohapatra, 1998; Akiba and Yamamoto, 2001; Beiu and Dăuş, 2015), even though the size of devices has shrunk dramatically. X-rays and disease diagnostics (Salvia and Lasher, 1990; Hsieh and Chen, 2004) are now joined by studies of biological systems at the cell scale (Beiu and Dăus, 2015). Nowadays, wireless sensor systems for security and communication pervade our lives; assessing their reliabilities is crucial (Makri and Psillakis, 1997; Habib et al., 2010; Cheng et al., 2016; Si et al., 2017; Nakamura et al., 2018; Liu, 2019; Malinowski, 2021). Pattern search systems (Aki and Hirano, 2004; Hsieh and Chen, 2004; Habib et al., 2010) are also a hot topic for AI techniques that are currently revolutionizing a large number of industrial sectors.

Various algorithms have been proposed to compute the reliability of the linear connected two-dimensional systems (Yamamoto and Miyakawa, 1995; Khamis, 1997; Yamamoto et al., 2008; Habib et al., 2010; Zhao et al., 2011; Nashwan, 2015; Nakamura et al., 2018), while keeping small values of r, s, and m. Extended surveys may be found in (Kuo and Zuo, 2003; Akiba et al., 2019). Let us also mention approaches using embedded Markov chains or Monte Carlo computations (Zhao et al., 2009; Zhao et al., 2012). The case $r = s = 2$ and $2 \leq m \leq 4$ was been revisited by (Malinowski, 2021), with algorithms of reduced complexity, in which the reliability is computed through nested recursions. Following that first effort, (Malinowski and Tanguy, 2022; Tanguy and Malinowski, 2023) showed that exact, analytical recurrence relations can be found for several cases of connected (r,2)-out-of-(m,n):F lattice systems with identical components, with m not so small anymore, and n arbitrarily large. Furthermore, they found that for large m and n, the reliability of the system has a simple asymptotic expression (Tanguy and Malinowski, 2023)

$$R_n \sim \delta_*(\gamma_*)^m(\chi_*)^n(\zeta_*)^{m\,n} \,, \tag{1}$$

where all the quantities are functions of the unreliability q of each component, as well as r and s.

Linear connected two-dimensional systems have not been the only architectures described in the literature. Cylindrical systems have also been considered by several authors (Boehme et al., 1992; Zuo, 1993; Preuss and Boehme, 1994; Yamamoto and Miyakawa, 1996; Khamis, 1997; Makri and Psillakis, 1997; Akiba and Yamamoto, 2001; Kuo and Zuo, 2003; Yamamoto and Akiba, 2005; Nashwan, 2015; Akiba et al., 2019; Nakamura et al., 2022; Nashwan, 2023). Even though the structure of the described systems is indeed cylindrical, the most prevalent adjective used is still "circular", as in the one-dimensional problem. In this case, the linear connected structure is folded back in one dimension. This folding procedure can be further extended to the two dimensions for a toroidal system, considered by (Nakamura et al., 2017; Akiba et al., 2019; Nakamura et al., 2022).

The purpose of this paper is to assess the probability of operation of circular (cylindrical) and toroidal (r,2)-out-of-(m,n):F lattice systems, thereby extending our previous results (Malinowski and Tanguy, 2022; Tanguy and Malinowski, 2023). The general aim is still to provide simple, analytical results that could give accurate results in essentially O(1) time. One also expects that in the large m and n limit, the behavior should be quite similar to that already obtained for linear connected systems.

The paper is organized as follows. In Section 2, we start with the circular case with $r = 3$, $s = 2$, and width $m = 4$ — the direction along which the folding is performed — keeping for n the direction in which the system can be arbitrarily large. Note that in a few papers, the notation m and n are reversed; it is nonetheless easy to translate the associated configurations. We describe in detail our method, and derive the recurrence relation allowing a complete solution of the problem. Section 3 is devoted to the same $r = 3$ and $s = 2$ (failure) configuration with different values of m, while n remains arbitrary. This will allow us to provide an asymptotic expression for the reliability of the CirCon/(3,2)-out-of-(m,n):F system. Section 4 considers various values of r, and generalize the results of the previous Sections. The toroidal systems are considered in Section 5, in which a different kind of behavior occurs, that can be explained very simply. We conclude by summing up our results and outline the direction of future work.

2. Detailed calculation for the $r = 3$, $s = 2$, $m = 4$ circular configuration

The aim of this section is to explain the existence of a recurrence relation between successive values of n. The gist of the method is given for $m = 4$ and is readily generalized. The case $m = 4$ has been chosen, since it has been detailed in (Tanguy and Malinowski, 2023) for the linear connected structure, making a comparison straightforward.

For a lattice of width $m = 4$, the successive objects are replaced by 1 if they are operating, and 0 if they are failed. The state of each row can then be seen as the binary representation of an integer k such that $0 \leq k \leq 15$ (in the general case, the bounds will be 0 and $2^m - 1$). The system will fail if there exists a 3 x 2 block of 0's when a new layer (row) is added. For such a configuration to occur, only thirteen ``transitions" are possible, as shown in Figure 1 (note the red 3 x 2 blocks of zeros). In the linear connected configuration, there were only seven such transitions.

One can merge the states described by integers 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, and 15 in a single state E (for "Else"). A new set of six states, namely $I = \{0, 1, 2, 4, 8, E\}$, is now under consideration. Let us denote by ${p_i}^{(n)}$ the probability that the n-layer lattice is still operating, provided that the last (nth) layer is described by the ith state, with $i \in I$. For instance, ${p_0}^{(n)} = \mathrm{Pr}_n(0)\, {p_E}^{(n-1)}$ (the only possibility), where $\mathrm{Pr}_n(0)$ is the probability of occurrence of state 0 in layer n. We thus have

$${p_0}^{(n)} = \mathrm{Pr}_n(0)\, {p_E}^{(n-1)} \tag{2}$$

$${p_1}^{(n)} = \mathrm{Pr}_n(1)\, ({p_2}^{(n-1)} + {p_4}^{(n-1)} + {p_8}^{(n-1)} + {p_E}^{(n-1)}) \tag{3}$$

$${p_2}^{(n)} = \mathrm{Pr}_n(2)\, ({p_1}^{(n-1)} + {p_4}^{(n-1)} + {p_8}^{(n-1)} + {p_E}^{(n-1)}) \tag{4}$$

$${p_4}^{(n)} = \mathrm{Pr}_n(4)\, ({p_1}^{(n-1)} + {p_2}^{(n-1)} + {p_8}^{(n-1)} + {p_E}^{(n-1)}) \tag{5}$$

$$p_8^{(n)} = \mathrm{Pr}_n(8)\ \left(p_1^{(n-1)} + p_2^{(n-1)} + p_4^{(n-1)} + p_E^{(n-1)}\right) \tag{6}$$

$$p_E^{(n)} = \mathrm{Pr}_n(E)\ \left(p_0^{(n-1)} + p_1^{(n-1)} + p_2^{(n-1)} + p_4^{(n-1)} + p_8^{(n-1)} + p_E^{(n-1)}\right) \tag{7}$$

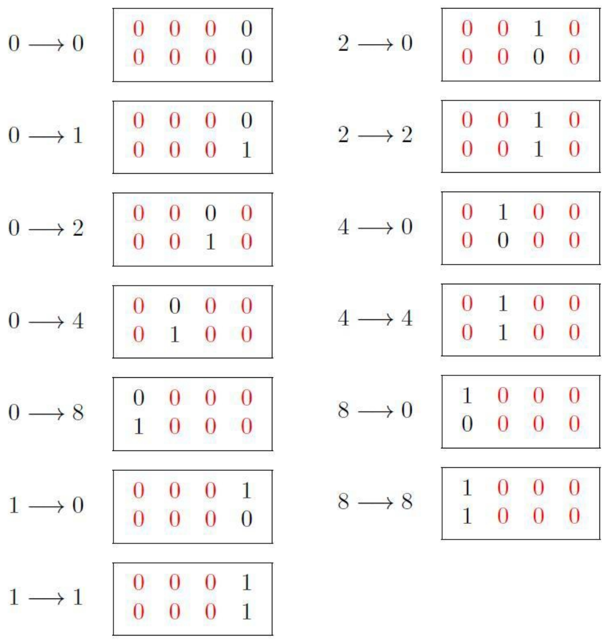

Fig. 1. Successive combinations leading to a system failure for a circular connected (3,2)-out-of-(4,*n*):F system.

For the sake of simplicity, we assume that $\mathrm{Pr}_n(i)$ does not depend on n; its simplified notation will be $p(i)$ from now on. The 6 x 6 transfer matrix M is what links the sets of $p_i^{(n-1)}$ and $p_i^{(n)}$, as apparent from (8):

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_4^{(n)} \\ p_8^{(n)} \\ p_E^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & p(0) \\ 0 & 0 & p(1) & p(1) & p(1) & p(1) \\ 0 & p(2) & 0 & p(2) & p(2) & p(2) \\ 0 & p(4) & p(4) & 0 & p(4) & p(4) \\ 0 & p(8) & p(8) & p(8) & 0 & p(8) \\ p(E) & p(E) & p(E) & p(E) & p(E) & p(E) \end{pmatrix} \begin{pmatrix} p_0^{(n-1)} \\ p_1^{(n-1)} \\ p_2^{(n-1)} \\ p_4^{(n-1)} \\ p_8^{(n-1)} \\ p_E^{(n-1)} \end{pmatrix} \tag{8}$$

When all elements are identical, with a probability of failure q, one gets $p(0) = q^4$, as well as $p(1) = p(2) = p(4) = p(8) = (1-q)\,q^3$, and $p(E) = 1 - 4\,q^3 + 3\,q^4$. The probability R_n of operation is obtained by summing the different contributions:

$$R_n = p_0^{(n)} + p_1^{(n)} + p_2^{(n)} + p_4^{(n)} + p_8^{(n)} + p_E^{(n)}\ , \tag{9}$$

which reduces to

$$R_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\ M^n \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{10}$$

The first values of R_n are easy to compute. After the trivial $R_0 = R_1 = 1$, one gets

$$R_2 = 1 - 4\,q^6 + 3\,q^8 \tag{11}$$

$$R_3 = 1 - 8\,q^6 + 6\,q^8 + 4\,q^9 + 12\,q^{10} - 24\,q^{11} + 9\,q^{12} \tag{12}$$

$$R_4 = 1 - 12\,q^6 + 9\,q^8 + 8\,q^9 + 24\,q^{10} - 48\,q^{11} + 30\,q^{12} - 24\,q^{13} + 12\,q^{14} \tag{13}$$

Because of the form of (10), R_n must obey a recurrence relation of order less than or equal to the dimension of M (six). The characteristic polynomial of M is

$$P(X) = (X + q^3 - q^4)^3\,(X^3 - (1 - q^3)X^2 - (1 - q)^2 q^3 (1 + 2q + 3q^2)X + 3(1 - q)^3 q^7 (1 + 2q + 3q^2)) \tag{14}$$

It turns out that only the polynomial of degree 3 is relevant for the recurrence relation, which can be written

$$R_n = (1 - q^3)\,R_{n-1} + (1 - q)^2 q^3 (1 + 2q + 3q^2)\,R_{n-2} - 3(1 - q)^3 q^7 (1 + 2q + 3q^2)\,R_{n-3} \tag{15}$$

We can deduce from (14) and the first values of R_n the generating function $G_4(z) = \sum_{n=0} R_n\ z^n$ (Stanley, 2011)

$$G_4(z) = \frac{1 + q^3\,z - 3\,(1-q)q^7\,z^2}{1 - (1-q^3)\,z - (1-q)^2 q^3 (1+2q+3q^2)\,z^2 + 3\,q^7 (1-q)^3 (1+2q+3q^2)\,z^3} \tag{16}$$

A partial fraction decomposition of (14) gives

$$G_4(z) = \frac{\alpha_0}{1-\zeta_0 z} + \frac{\alpha_1}{1-\zeta_2 z} + \frac{\alpha_2}{1-\zeta_2 z} \tag{17}$$

from which one deduces

$$R_n = \alpha_0\,\zeta_0{}^n + \alpha_1\,\zeta_1{}^n + \alpha_2\,\zeta_2{}^n \tag{18}$$

where, after some work,

$$\zeta_k = \frac{1}{3}\ \left(1 - q^3\ +\ 2\sqrt{1 + q^3 - 11\,q^6 + 9\,q^7}\,\cos\left(\frac{1}{3}(2\,k\,\pi\ +\ \phi)\right)\right) \qquad (k \in \{0,1,2\}) \tag{19}$$

$$\phi = \arccos\left(\frac{2 + 3\,q^3 - 39\,q^6 - 54\,q^7 + 81\,q^8 + 34\,q^9 + 297\,q^{10} - 567\,q^{11} + 243\,q^{12}}{2\,(1 + q^3 - 11\,q^6 + 9\,q^7)^{3/2}}\right) \tag{20}$$

$$\alpha_k = \frac{\zeta_k{}^2 + q^3 \zeta_k - 3\,(1-q)\,q^7}{3\,\zeta_k{}^2 - 2\,(1-q^3)\,\zeta_k - (q^3 - 4\,q^6 + 3\,q^7)} \tag{21}$$

The variation of the ζ_k with q are displayed in Figure 2. Only one of them (ζ_0) is close to 1 when q nears 0.

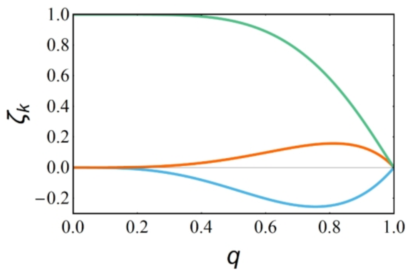

Fig. 2. Variation with q of the three eigenvalues when $m = 4$. $\zeta_0\ \equiv\ \zeta_+$ is represented in green.

When n is large, even for moderate values of q, (18) obviously reduces to a power-law behavior of R_n

$$R_n \approx \alpha_0\,\zeta_0{}^n \tag{22}$$

Such a behavior was already observed in the linear connected (3,2)-out-of-(4,n):F lattice system (Tanguy and Malinowski, 2023), with slightly different values of the prevailing eigenvalue $\zeta_0\ \equiv\ \zeta_+$, since the recurrence relation is different. Here again, the computation of R_n can be performed in essentially O(1) time.

Equation (22) can be rewritten as, with ζ_+ denoting from now on the prevailing eigenvalue,

$$\ln R_n \approx \ln \alpha_+ + n \ln \zeta_+ \tag{23}$$

Such an expression will be useful in the next section, when an analytical determination of the prevailing root ζ_+ is not possible anymore when the order of the recurrence becomes too large. The Taylor expansions of $\ln \zeta_+$ and $\ln \alpha_+$ can be obtained through

$$\ln \zeta_+\ \approx\ \ln R_n\ -\ \ln R_{n-1} \tag{24}$$

$$\ln \alpha_+\ \approx n \ln R_{n-1} - (n - 1) \ln R_n \tag{25}$$

3. Results for the $r = 3$, $s = 2$, $m \geq 5$ circular configurations

The above method has been used for $5 \leq m \leq 13$, and the results are similar. The number of eigenvalues, which are all real, increases with m: 4, 4, 4, 6, 8, 10, 11, 16, and 21, respectively. In the case $m = 3$, there are only two of them. The variation of the ζ_k with q are displayed in Figure 3 for $m = 13$. A single one of them (ζ_+) is close to 1 when q nears 0.

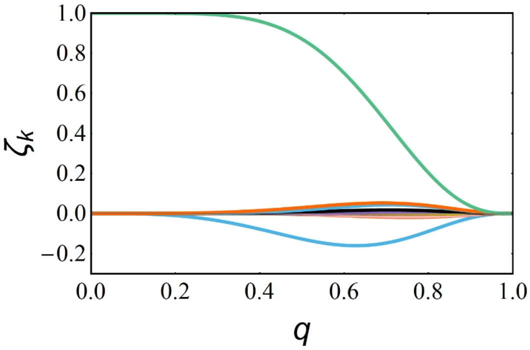

Fig. 3. Variation with q of the 21 eigenvalues when $m = 13$. ${\zeta_+}^{(13)}$ is represented in green.

It is possible to compute the Taylor expansion for small q of $\ln {\zeta_+}^{(m)}$ and $\ln {\alpha_+}^{(m)}$, for the abovementioned values of m by using (24) and (25). In (Tanguy and Malinowski, 2023), it was shown that for linear connected (r,2)-out-of-(m,n):F systems,

$${\zeta_+}^{(m)} \approx \chi_* \ {\zeta_*}^m \tag{26}$$

$${\alpha_+}^{(m)} \approx \delta_* \ {\gamma_*}^m \tag{27}$$

and Taylor expansions were provided for the four quantities ζ_*, χ_*, γ_*, and δ_*. Here, however, when *cylindrical* systems are considered, it appears that χ_* and δ_* are equal to 1, while ζ_* and γ_* have the same Taylor expansions as in ***linear*** connected systems. Finally,

$$R_n \approx (\gamma_*)^m \, (\zeta_*)^{m\,n} \tag{28}$$

This result is not surprising since when m and n are large, the "boundary conditions" linked to circular/cylindrical systems play a vanishing role.

4. Results for different values of r (for $s = 2$) in circular configurations

In the preceding Section, we obtained a simple asymptotic expression in (28) for the probability of operation of a (3,2)-out-of-(m,n):F system. Each value of ζ_* and γ_* depends explicitly on the probability of failure q. These parameters also depend ***implicitly*** on the specific values $r = 3$ and $s = 2$. Our aim in this section is to report what occurs when r varies.

We have performed the same calculations and processing of the results for $r = 2$ and $r = 4$, with m as high as 14. While the degrees of the recurrences may vary, the general behavior of (28) is found again, with slightly modified expressions for ζ_* and γ_*.

In the case $r = s = 2$, for $m = 4$ the recurrence relation reads

$$\begin{aligned} R_n = {} & (1-q)\,(1+q-q^3)\,R_{n-1} + (1-q)^2\,q^2\,(1+q)\,(1+2q-2q^2)\,R_{n-2} \\ & -(1-q)^3\,q^5\,(1+2q-q^2)\,(1+q-q^2)\,R_{n-3} - (1-q)^5\,q^9\,(1+2q-q^2)\,R_{n-4} \end{aligned} \tag{29}$$

with again $R_0 = R_1 = 1$, and

$$R_2 = 1 - 4\,q^4 + 4\,q^6 - q^8 \tag{30}$$

$$R_3 = 1 - 8\,q^4 + 12\,q^6 + 8\,q^7 - 14\,q^8 - 12\,q^9 + 20\,q^{10} - 8\,q^{11} + q^{12} \tag{31}$$

where R_3 is in full agreement with previously published results (Nashwan, 2015, 2023). The derived expressions also agree with the results given by (Yamamoto and Miyakawa, 1996) for larger values of *n*, up to 50. These authors also consider in the same article the case $m = 10$, and provide numerical evaluations for $n = 10$ and $n = 50$, with which our derivation is in complete agreement. Note that for $m = 10$ the recurrence relation is of order 20 and could not reasonably be reproduced here.
Taking into account the results obtained for various values of *r*, we recover the results of (Tanguy and Malinowski, 2023) when linear connected systems were considered:

$$\ln\ \zeta_* = -q^{2r}(1-q^2) + q^{3r}(1+2\,q-2\,q^2-q^3) \\ +q^{4r}\left(-\frac{6\,r+5}{2} - 6\,q + (6\,r+3)q^2 + 6\,q^3 - \frac{6\,r+1}{2}q^4\right) + \cdots \quad (32)$$

$$\ln \gamma_* = q^{2r}(1-q^2) + q^{3r}(-2-4q+4q^2+2q^3) \\ +q^{4r}(\frac{1}{2}(11+10\,r) + 16\,q - 5\,(1+2\,r)q^2 - 16\,q^3 + \frac{1}{2}(10\,r-1)\,q^4) + \ldots \quad (33)$$

These expressions, combined with (28), could be useful for the assessment of the probability of operation of large circular systems.

5. Toroidal configurations

We now consider toroidal systems that can describe equipment such as particle accelerators, storage tanks, or even the architecture of interconnection networks (Nakamura et al., 2017; Akiba et al., 2019; Nakamura et al., 2022). Two questions may be asked: (i) How does one proceed from the circular configuration? (ii) Is it possible to derive an asymptotic expression for the probability of operation when *m* and *n* are large?

The answer to the first question is relatively easy. One must start with the circular two-dimensional configuration, in which the lattice has been folded (in the present work, in the dimension associated with *m*). One can therefore use the same approach as in Section 2. If one considers the *n*th row, the calculation of must take two facts into account: there cannot be a failure structure in rows *n*-1 and *n* (as in Section 2), and since the *n*th row is the final one, there cannot be a failure pattern with rows *n* and 1. Consequently, one must consider separately the different possibilities associated with each first row.

If the first row is described by 0, the second one must be represented by *E*. So must be the third row, the fourth one, etc. The probability of operation of the toroidal system when the first row is 0 is therefore equal to $p(0)p(E)^{n-1}$.

If the first row is described by *E*, the second one can be arbitrary and we recover the standard problem of a circular connected (*r*,2)-out-of-(*m*,*n*-1):F problem, which has been addressed in the preceding Sections.

The remaining cases must be treated differently. Considering the configuration $r = 3, s = 2, m = 4$ of Section 2, if the state of the first row is represented par 1, it is clear that the following row cannot be represented by 0 or by 1, because a succession of two such rows would imply a failure. The consequence is very simple: only 2, 4, 8, and *E* may represent the next rows. The new transfer matrix M_1 is deduced from the one appearing in (8) by multiplying it by the matrix associated to the projection on the space ``orthogonal'' to 0 and 1:

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & p(0) \\ 0 & 0 & p(1) & p(1) & p(1) & p(1) \\ 0 & p(2) & 0 & p(2) & p(2) & p(2) \\ 0 & p(4) & p(4) & 0 & p(4) & p(4) \\ 0 & p(8) & p(8) & p(8) & 0 & p(8) \\ p(E) & p(E) & p(E) & p(E) & p(E) & p(E) \end{pmatrix} \quad (34)$$

The contribution to the probability of operation R_n(first row: 1) is then given by

$$R_n(\text{first row: } 1) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} (M_1)^{n-1} \begin{pmatrix} 0 \\ p(1) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (35)$$

A similar treatment may be used for the remaining states 2, 4, and 8. One must finally sum all the contributions to obtain the total probability of operation R_n.

This derivation shows that the probability of operation of a toroidal system is more complicated than its linear or circular counterparts. One expects, for each initial row, a leading eigenvalue that goes to 1 when q goes to zero. This is demonstrated by the expression of the generating function $G_4(z)$ of R_n.

$$G_4(z) = -\frac{1}{1-4\,q^3+3\,q^4} + \frac{q^4}{(1-4\,q^3+3\,q^4)(1-(1-4\,q^3+3q^4)\,z)} + \frac{4\,q^3(1-2\,q^3(1-q)\,z)}{(1-q)\,(1+2q+3q^2)\,(1-(1-2\,q^3+q^4)\,z-q^3(1-q)^3(1+2\,q+3\,q^2)\,z^2)} + \\ + \frac{1-3q^3(1-q)\,z}{1-(1-q^3)\,z-q^3(1-4\,q^3+3\,q^4)\,z^2+3\,(1-q)^3q^7(1+2\,q+3\,q^2)\,z^3} \quad (36)$$

The second term of (36) corresponds to the contribution of a first row represented by 0, the third term by the rows represented by 1, 2, 4, and 8, which are equivalent, while the last term corresponds to the first row described by *E*. Note that the first term ensures that $R_1 = 1$, as it should. The leading ζ_+ are displayed in Figure 4, the prevailing one (in green) is associated to *E* in the first row.

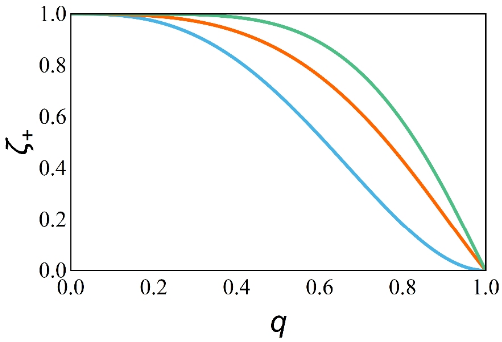

Fig. 4. Variation with q of the leading ζ_+ for the toroidal $r = 3, s = 2, m = 4$ system.

The toroidal (2,2)-out-of-(3,3) has also been studied (Nakamura et al., 2022). For identical elements, one finds

$$R_n(2,2;3,3) = 1 - 9\,q^4 + 12\,q^6 + 18\,q^7 - 36\,q^8 + 14\,q^9 \quad (37)$$

(Nakamura et al., 2022) have also considered other values of m and n (keeping $r = s = 2$), with alternate unavailabilities of the system's elements. A comparison of their results with ours is not currently possible.

For the sake of completeness, the leading eigenvalues for the leading ζ_+ for the toroidal $r = 2, s = 2, m = 3$ system are also displayed in Figure 5. They may compete but for large n, the green one will prevail.

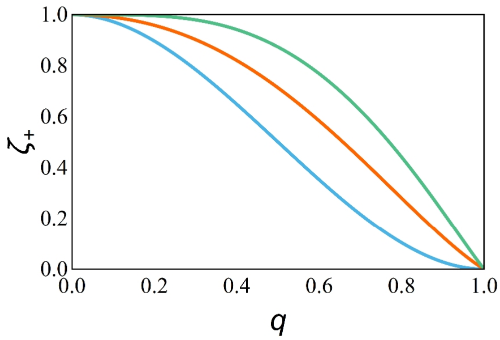

Fig. 5. Variation with q of the leading ζ_+ for the toroidal $r = 2, s = 2, m = 3$ system.

5.1. Asymptotic expansion for the toroidal case

From the previous description of the different possibilities in the toroidal configuration and (28), one can deduce that the asymptotic behaviour of ${R_n}^{(T)}$ will be driven by

$$R_n^{(T)} \approx p(E)\ (\gamma_*)^m\ (\zeta_*)^{m(n-1)} \tag{38}$$

where $p(E)$ is the probability that there is no possibility of having a r x 2 block of failed components if one adds a second row. This means that there are no block of r consecutive zeros in the first row. Therefore, $p(E)$ is nothing but the probability of a one-dimensional CirCon/r-out-of-m:F. Since (analytical expressions of $p(E)$ exist for the cases $r = 2$ and $r = 3$) for large m,

$$p(E) \approx (\xi_*)^m \tag{39}$$

one can conclude that

$$R_n^{(T)} \approx (\frac{\xi_* \gamma_*}{\zeta_*})^m\ (\zeta_*)^{mn} \tag{40}$$

This asymptotic expansion has been checked in our numerical simulations, for various values of r and m. The form of the expression is similar to (28), with a different prefactor to the mth power. Note that Taylor expansions of all the parameters in (40) are known to a very large order (Tanguy and Malinowski, 2023). Still, the convergence to (40) is much slower than in the linear or circular configurations.

6. Conclusion and outlook

In the present work we have adapted to circular and toroidal two-dimensional systems the method designed for linear-connected (r,2)-out-of-(m,n):F lattice systems (Malinowski and Tanguy, 2022; Tanguy and Malinowski, 2023). While the circular case only marginally differs from the linear one, the toroidal case is more involved. It is nevertheless possible to provide an asymptotic expansion in both cases for large values of m and n (see (28) and (40)). Further work will be devoted to the study of the probability of operation of such systems when s is larger than 2, and when the probabilities of failure of the elements are not identical, as in (Akiba et al., 2019; Nakamura et al., 2017, 2022).

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