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Estimation Of Weibull Distribution Parameters Based On Repeated Minimal Repairs: Method Accuracy

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Abstract

This paper extends the results of (Malinowski, 2022), where a new non-standard method of estimating the shape (alpha) and scale (lambda) parameters of the Weibull distribution is proposed. The method uses n identical, repairable, mutually independent sample units and proceeds as follows: First, each item undergoes m–1 failures followed by minimal repairs and is discarded after the m-th failure. The resulting times-to-failure compose n independent samples, each of size m, where the times within one sample are neither independent nor identically distributed. Second, the obtained data are used to compute n MLE estimates of alpha and lambda, where the j-th pair of estimates are based on the times-to-failure of the j-th item, $j = 1,\ldots,n$. Third, the mean value of the above n estimates is computed for each parameter, to obtain more accurate estimates. The main results of (Malinowski, 2022) are the closed-form expressions for the considered MLE estimators and their biases. It should be noted here that the generally known MLE estimator of the shape parameter of the Weibull distribution, based on IID sampling, is obtained from an equation that cannot be solved analytically. However, the accuracy of the estimators obtained in (Malinowski, 2022) was not established, because no formulas for their variances could be found. This difficulty has been overcome in the present paper, where the respective formulas are derived and then used in the analysis of confidence intervals for alpha and lambda.

Keywords: Weibull distribution, minimal repair, maximum likelihood estimation, gamma distribution, cumulants, variance, confidence interval

1. Introduction

In (Malinowski, 2022) a new method of estimating the scale and shape parameters of the Weibull distribution is proposed. As generally known, this distribution has the following PDF:

$$f(t) = \alpha\lambda^{\alpha}(t)^{\alpha-1}\exp[-(\lambda t)^{\alpha}] \tag{1}$$

where α and λ are the shape and scale parameters respectively. The issue of estimating these parameters using various estimation methods has been widely studied by multiple statisticians (Dodson, 2006; Chikr el-Mezouar, 2010; Nielsen, 2011; Osarumwense and Rose, 2014; Almazah and Ismail, 2021). A recent survey of relevant literature can be found in (Jokiel-Rokita and Piątek, 2022).The Weibull probabilistic model has diverse practical applications, as demonstrated, inter alia, in (Evans et al., 2019; Lei, 2008; Wu et al., 2021). Although the topic has been extensively researched, a new approach inspired by reliability theory is presented in this paper. It is a well-known fact that the time-to-failure (TTF) of many technical devices (or their components) is a Weibull distributed random variable. Therefore, in order to estimate its parameters, the standard method is to measure the IID times-to-failure of a number of non-repairable test items, and calculate the required estimates from the values of the random sample. Such a procedure is followed in (Alizadeh et al., 2015; Almazah and Ismail, 2021; Wu et al., 2021), to name a few. Sometimes, due to restrictions imposed on the sampling time, only censored data are available. Weibull estimation with such data is discussed in Alkutubi and Ali, 2011. Regardless whether the sample is complete or censored, the standard approach has one essential disadvantage – if failed objects are no longer usable then a large number of test items are needed in order to achieve high estimation accuracy, which may lead to unacceptable cost. However, if the test items are repairable, then this cost can be reduced by applying a method presented further in this paper.

The proposed method uses n identical, repairable, mutually independent sample units, each of which undergoes m–1 minimal repairs following its first m–1 failures (m≥2) and is discarded after the m-th failure. The resulting times-to-failure can be arranged in a matrix $[t_{ij}]_{i=1,\dots,m;\, j=1,\dots,n}$ where t_{ij} is the i-th time-to-failure of the j-th item, i.e. the time from the completion of its (i–1)-th minimal repair to the moment of its i-th failure. We assume that the 0-th repair precedes the start of operation of a new item. For each j=1,…,n the sample times $t_{1,j},\dots,t_{m,j}$ are realizations of the random variables $T_1,\dots,T_m$, where T_i is the random i-th time-to-failure of a sample item (note the difference between a random variable and its realization which is a fixed value). In order not to complicate the notation, the index j will be omitted, thus the sample times-to-failure of any of n items will be denoted by $t_1,\dots,t_m$.

Each parameter α or λ is approximated by two estimators. The first one, named m-sample estimator, is a not-so-accurate MLE based on a sample of m times-to-failure of one item, while the second one, named $n{\cdot}m$-sample estimator, is the mean value of the n IID realizations of the first estimator, thus ensuring much greater accuracy. The obtained estimators are biased and in (Malinowski, 2022) simple analytical formulas for their biases were found. However, the estimators' accuracy could not be established due to lack of formulas for their variances.

As far as minimal repairs are concerned, there exist many reliability models with this type of repair. For various specific models and literature surveys see (Aven and Jensen, 2000; Tadj et al., 2011; Knopik and Migawa, 2019; Navarro et al., 2019; de Jonge and Scarf, 2020; Rebaiaia and Ait-Kadi, 2021; Liu and Wang, 2022). A practical example of a minimal repair is given in Cha, 2005.

The current paper is organized as follows. Section 2 summarizes the main results of (Malinowski, 2022), providing the basis for further argument. In Section 3 we formulate an auxiliary lemma used in Sections 4 and 5, where new closed-form exact expressions for the biases and variances of the estimators of α and λ are found. Since the exact expressions derived in Sections 4 and 5 include the estimated parameter α, the approximate expressions with α replaced by its estimator, suitable for numerical computations, are given in Section 6. In Section 7 the confidence intervals for the considered parameters are discussed, where the main result are the formulas for n guaranteeing that the interval of a given width contains the respective parameter with a given confidence level. Finally, Section 8 contains concluding remarks and prospects for future work.

2. A summary of the previous work

The derivations carried out in (Malinowski 2022) led to the following likelihood function based on m successive times-to-failure of one item, i.e. on one m-sample:

$$f^{(m)}(t_1,\dots,t_m|\alpha,\lambda) = \prod_{i=1}^{m-1} r(s_i)\, f(s_m) = [\alpha^m \lambda^{\alpha m} \prod_{i=1}^{m} (s_i)^{\alpha-1}] \times \exp\,[-(\lambda)^\alpha (s_m)^\alpha] \tag{2}$$

Here, $f(\cdot)$ is the Weibull PDF given by (1), $r(\cdot)$ is the failure rate function, i.e.

$$r(t) = \frac{f(t)}{1-F(t)} \tag{3}$$

where F is the Weibull CDF, and $s_i = t_1 + \dots + t_i$, i=1,…,m. By equating the first derivatives of $\ln[f^{(m)}]$ with respect to α and λ, the following m-sample estimators of these parameters, denoted as $\hat{\alpha}$ and $\hat{\lambda}$, have been obtained:

$$\hat{\alpha} = \left[\ln(s_m) - \frac{1}{m}\sum_{i=1}^{m}\ln(s_i)\right]^{-1}, \quad \hat{\lambda} = m^{1/\hat{\alpha}}(s_m)^{-1} \tag{4}$$

Let us note that the expression for $\hat{\lambda}$ contains $\hat{\alpha}$, which can be substituted according to the left part of (4), yielding $\hat{\lambda}$ as a function of m and $s_1,\dots,s_m$. For reasons of analytical tractability, explained in more detail in (Malinowski, 2022), it is more convenient to estimate $1/\alpha$ and $\ln(\lambda)$ rather than directly α and λ, the respective m-sample estimators being $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$. From (4) we easily obtain the following formulas:

$$\frac{1}{\hat{\alpha}} = \frac{m-1}{m}\ln(s_m) - \frac{1}{m}\sum_{i=1}^{m-1}\ln(s_i) \tag{5}$$

and

$$\ln(\hat{\lambda}) = \frac{1}{\hat{\alpha}}\ln(m) - \ln(s_m) = \left[\frac{m-1}{m}\ln(m) - 1\right]\ln(s_m) - \frac{\ln(m)}{m}\sum_{i=1}^{m-1}\ln(s_i) \tag{6}$$

As mentioned earlier, α and λ, or rather $1/\alpha$ and $\ln(\lambda)$, are better approximated by $n{\cdot}m$-sample estimators than by m-sample ones. The former, denoted by $\hat{A}$ and $\hat{\Lambda}$, are defined as follows:

$$\hat{A} = \left(\frac{1}{\hat{\alpha}_1} + \cdots + \frac{1}{\hat{\alpha}_n}\right)/n \tag{7}$$

$$\hat{\Lambda} = \left[\ln(\hat{\lambda}_1) + \cdots + \ln(\hat{\lambda}_n)\right]/n \tag{8}$$

where $\hat{\alpha}_j$ and $\hat{\lambda}_j$ are obtained from the successive times-to-failure of the j-th item by applying (4). Clearly,

$$E(\hat{A}) = E(1/\hat{\alpha}),\ \ E(\hat{\Lambda}) = E\left[\ln(\hat{\lambda})\right] \tag{9}$$

and, since the sequences of times-to-failure are independent between items,

$$Var(\hat{A}) = Var(1/\hat{\alpha})/n, \quad Var(\hat{\Lambda}) = Var\left[\ln(\hat{\lambda})\right]/n \tag{10}$$

The biases of the m-sample estimators of $1/\alpha$ and $\ln(\lambda)$ are equal to $1/\alpha - E(1/\hat{\alpha})$ and $\ln(\lambda) - E\left[\ln(\hat{\lambda})\right]$, while the biases of the respective $n{\cdot}m$-sample ones are equal to $1/\alpha - E(\hat{A})$ and $\ln(\lambda) - E(\hat{\Lambda})$. As follows from (9) and (10), $n{\cdot}m$-sample estimators have the same biases as m-sample ones, but are n times more accurate if the accuracy of an estimator is measured by its variance.

A natural question arises – how good are the estimators defined by (5) - (8)? The answer requires the knowledge of the biases and variances of $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$. In (Malinowski 2022) the following formulas were derived for the respective biases:

$$1/\alpha - E(1/\hat{\alpha}) = \frac{1}{m-1}E(1/\hat{\alpha}) \tag{11}$$

and

$$\ln(\lambda) - E\left[\ln(\hat{\lambda})\right] = \frac{m}{m-1}E(1/\hat{\alpha}) \times \left[\sum_{i=1}^{m-1}\frac{1}{i} - \frac{m-1}{m}\ln(m) - \gamma\right] \tag{12}$$

where γ is the Euler-Mascheroni constant defined below

$$\gamma = \lim_{m\to\infty}\left[\sum_{j=1}^{m}\frac{1}{j} - \ln(m)\right] \cong 0.5772 \tag{13}$$

Now it only remains to compute $E(1/\hat{\alpha})$ to obtain the biases given by (11) and (12). According to (11), it holds that

$$E(1/\hat{\alpha}) = (m-1)(m\alpha)^{-1} \tag{14}$$

hence the bias of $1/\hat{\alpha}$ is equal to $(m\alpha)^{-1}$. Still, we do not have the exact value of α, as it is a parameter to be estimated. But, in view of (7) and the law of large numbers, $E(1/\hat{\alpha})$ can be approximated by $\hat{A}$, i.e. the $n{\cdot}m$-sample estimator of $1/\alpha$, where the accuracy of the approximation increases with n. Therefore, (11) with $\hat{A}$ instead of $E(1/\hat{\alpha})$ on the right-hand side can be used to compute the near-exact bias of $1/\hat{\alpha}$.

Similarly, we can express the bias of $\ln(\hat{\lambda})$ as a function of α and m by substituting $E(1/\hat{\alpha})$ with $(m-1)(m\alpha)^{-1}$ in (12). It is somewhat unexpected that this bias does not depend on λ. In turn, the near-exact bias of $\ln(\hat{\lambda})$ can be computed by substituting $E(1/\hat{\alpha})$ with $\hat{A}$ in (12).

Let us note that (11) – (13) imply that $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$ are asymptotically unbiased as $m\to\infty$. This fact is of little practical significance, because a sample item often becomes unusable after several repairs.

The formulas for the biases of $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$, i.e. (11) and (12), are the main result of (Malinowski, 2022). However, the accuracy of these estimators and their biases could not be assessed, because no closed-form expressions for the variances of $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$ were found. This shortcoming is improved in the current paper, where such expressions are given. For this purpose, we prove that $1/\hat{\alpha}$ is gamma distributed, hence its variance is given by a simple formula, then we find the cumulants of $\ln(\hat{\lambda})$ and use a well-known fact that the variance of a random variable is equal to its second cumulant. Moreover, having obtained the variances, we analyze the confidence intervals for the parameters $1/\alpha$ and $\ln(\lambda)$.

3. An auxiliary lemma

In the next two sections we will need the following auxiliary lemma:

Lemma 1
Let $u_1,\ldots,u_m$ be arbitrary positive real numbers and $v_i = u_1+\ldots+u_i$, $i=1,\ldots,m$. Then

$$E\left(S_1^{u_1}S_2^{u_2}\ldots S_m^{u_m}\right) = \left[\lambda^{v_m}\prod_{i=1}^{m-1}\left(\frac{v_i}{\alpha}+i\right)\right]^{-1}\Gamma\left(\frac{v_m}{\alpha}+m\right) \tag{15}$$

Proof:
Note that $S_1 < S_2 < \ldots < S_m$. Hence, from (2) we have:

$$E\left(S_1^{u_1}S_2^{u_2}\dots S_m^{u_m}\right) = \int_{0<s_1<\dots<s_m<\infty} f^{(m)}(s_1,\dots,s_m|\alpha,\lambda)ds_1\dots ds_m$$

$$= \int_0^\infty ds_m f(s_m)s_m^{u_m}\int_0^{s_m} ds_{m-1}r(s_{m-1})s_{m-1}^{u_{m-1}}\dots\int_0^{s_3} ds_2 r(s_2)s_2^{u_2}\int_0^{s_2} ds_1 r(s_1)s_1^{u_1} \tag{16}$$

For greater clarity, the differentials $ds_m,\dots,ds_1$ are placed before the integrands. Since $r(s)$ is the failure rate of the Weibull distribution, it holds that

$$\int_0^z r(s)s^u ds = \int_0^z \alpha\lambda^\alpha s^{\alpha-1}s^u ds = \alpha\lambda^\alpha\frac{1}{u+\alpha}z^{u+\alpha} \tag{17}$$

Based on (17), it can be proved by induction that

$$\int_0^{s_m} ds_{m-1}r(s_{m-1})s_{m-1}^{u_{m-1}}\dots\int_0^{s_3} ds_2 r(s_2)s_2^{u_2}\int_0^{s_2} ds_1 r(s_1)s_1^{u_1} = \lambda^{(m-1)\alpha}\left[\prod_{i=1}^{m-1}\left(\frac{v_i}{\alpha}+i\right)\right]^{-1}s_m^{v_{m-1}+(m-1)\alpha} \tag{18}$$

Using the above equality in (16) we obtain:

$$E\left(S_1^{u_1}S_2^{u_2}\dots S_m^{u_m}\right)$$

$$= \lambda^{(m-1)\alpha}\left[\prod_{i=1}^{m-1}\left(\frac{v_i}{\alpha}+i\right)\right]^{-1}\int_0^\infty ds_m\alpha\lambda^\alpha s_m^{\alpha-1}\exp[-(\lambda s_m)^\alpha]\,s_m^{v_m+(m-1)\alpha}$$

$$= \lambda^{(m-1)\alpha}\left[\prod_{i=1}^{m-1}\left(\frac{v_i}{\alpha}+i\right)\right]^{-1}\lambda^{-v_m-(m-1)\alpha}\Gamma(1+\frac{v_m}{\alpha}+m-1)$$

$$= \left[\lambda^{v_m}\prod_{i=1}^{m-1}\left(\frac{v_i}{\alpha}+i\right)\right]^{-1}\Gamma\left(\frac{v_m}{\alpha}+m\right) \tag{19}$$

This completes the proof. The penultimate equality is a consequence of the fact that

$$\int_0^\infty x^k f(x)dx = \lambda^{-k}\Gamma\left(1+\frac{k}{\alpha}\right) \tag{20}$$

where $f(x)$ is the Weibull distribution's PDF and k is a positive number (not necessarily integer).

4. Finding the distribution of $1/\hat{\alpha}$, its expected value and variance

Let the PDF of $1/\hat{\alpha}$ be denoted as $h(x)$, $x\geq 0$. We will find $h(x)$ by first computing and then inverting its Laplace transform. It holds that

$$\mathcal{L}\{h(x)\}(z) = \int_0^\infty h(x)\exp(-zx)\,dx = E\left(e^{-z/\hat{\alpha}}\right) \tag{21}$$

From (5) we have:

$$e^{-z/\hat{\alpha}} = s_m^{-z\frac{m-1}{m}}\times\prod_{i=1}^{m-1}s_i^{\left(\frac{z}{m}\right)} \tag{22}$$

The above expression has the form required by Lemma 1 with

$$u_i = \frac{z}{m},\ i = 1,\dots,m-1$$

$$u_m = -z\frac{m-1}{m} \tag{23}$$

Equation (23) yields:

$$v_i = \frac{iz}{m},\ i = 1,\dots,m-1$$

$$v_m = 0 \tag{24}$$

Substituting the above values of $v_1,\dots,v_m$ in (15), we obtain:

$$E\left(e^{-z/\hat{\alpha}}\right) = \left[\prod_{i=1}^{m-1}\left(\frac{iz}{m\alpha}+i\right)\right]^{-1}\Gamma(m) = \left(\frac{z}{m\alpha}+1\right)^{1-m} \tag{25}$$

The last equality in (25) is a consequence of the fact that $\Gamma(m)=(m-1)!$. From the Laplace transform properties we know that

$$\mathcal{L}\{x^n e^{-bt}\}(z) = \frac{n!}{(b+z)^{n+1}} = \frac{n!}{b^{n+1}}\left(1+\frac{z}{b}\right)^{-n-1} \tag{26}$$

Therefore

$$\mathcal{L}\left\{\frac{b^{n+1}}{n!}x^n e^{-bx}\right\}(z) = \left(1+\frac{z}{b}\right)^{-n-1} \tag{27}$$

If we put $b=m\alpha$ and $n=m-2$, then (27) converts to

$$\mathcal{L}\left\{\frac{(m\alpha)^{m-1}}{(m-2)!}x^{m-2}e^{-m\alpha x}\right\}(z) = \left(1+\frac{z}{m\alpha}\right)^{1-m} \tag{28}$$

Equations (21), (25) and (28) yield the following expression for $h(x)$, i.e. the PDF of $1/\hat{\alpha}$:

$$h(x) = \frac{(m\alpha)^{m-1}}{(m-2)!}x^{m-2}e^{-m\alpha x} \tag{29}$$

Hence, $1/\hat{\alpha}$ is gamma distributed with $m-1$ and $m\alpha$ as the shape and scale parameters. Basic properties of the gamma distribution yield that

$$E(1/\hat{\alpha}) = \frac{m-1}{m\alpha}, \quad Var(1/\hat{\alpha}) = \frac{m-1}{(m\alpha)^2} \tag{30}$$

Thus, the expected value and variance of $1/\hat{\alpha}$ are simple functions of α and m. Let us note that the first equality in (30) is already known from the Introduction (see the statement under (14)).

5. Computing the variance of $\ln(\hat{\lambda})$ using its cumulants

We will first compute the cumulant generating function (CGF) of $\ln(\hat{\lambda})$, i.e. the function $K_\lambda(z)$ defined as follows:

$$K_\lambda(z) = \ln\left[E\left(e^{z\ln(\hat{\lambda})}\right)\right] \tag{31}$$

From (6) we obtain:

$$e^{z\ln(\hat{\lambda})} = s_m^{z\left[\frac{m-1}{m}\ln(m)-1\right]} \times \prod_{i=1}^{m-1} s_i^{-z\frac{\ln(m)}{m}} \tag{32}$$

The above expression has the form required by Lemma 1 with

$$u_i = -z\frac{\ln(m)}{m},\ i = 1,\dots,m-1$$

$$u_m = z\left[\frac{m-1}{m}\ln(m) - 1\right] \tag{33}$$

Equation (33) yields:

$$v_i = -z\frac{\ln(m)}{m}i,\ i = 1,\dots,m-1$$

$$v_m = -z \tag{34}$$

Substituting the above values of $v_1,\dots,v_m$ in (15), we obtain:

$$E\left(e^{z\ln(\hat{\lambda})}\right) = \Gamma\left(m-\frac{z}{\alpha}\right)\left[(m-1)!\,\lambda^{-z}\left(1-z\frac{\ln(m)}{m\alpha}\right)^{m-1}\right]^{-1} \tag{35}$$

In view of (31) and (35), the CGF of $\ln(\hat{\lambda})$ is given by

$$K_\lambda(z) = \ln\left[\Gamma\left(m-\frac{z}{\alpha}\right)\right] - \ln[(m-1)!] + z\ln(\lambda) - (m-1)\ln\left[1-z\frac{\ln(m)}{m\alpha}\right] \tag{36}$$

The successive cumulants of $\ln(\hat{\lambda})$ are obtained by computing the respective derivatives of $K_\lambda(z)$ at $z = 0$. Let us note that $z\cdot\ln(m)/m\alpha < 1$ and $m-(z/\alpha) > 0$ for $z < \alpha$, thus the expression on the right-hand side of (36) is correct for sufficiently small z, which guarantees the existence of these derivatives. In view of (36) we have:

$$\frac{dK_\lambda(z)}{dz} = -\frac{1}{\alpha}\Psi\left(m-\frac{z}{\alpha}\right) + \ln(\lambda) + (m-1)\frac{\ln(m)}{m\alpha}\left[1-z\frac{\ln(m)}{m\alpha}\right]^{-1} \tag{37}$$

and

$$\frac{d^qK_\lambda(z)}{dz^q} = (-1)^q\frac{1}{\alpha^q}\Psi^{(q-1)}\left(m-\frac{z}{\alpha}\right) + (m-1)\left[\frac{\ln(m)}{m\alpha}\right]^q(q-1)!\left[1-z\frac{\ln(m)}{m\alpha}\right]^{-q} \tag{38}$$

where Ψ is the digamma function, i.e. $\Psi(z)=d\ln[\Gamma(z)]/dz$ and $q\geq 2$. Equation (38) can easily be proved by induction. A simple consequence of (37) and (38) is that

$$\kappa_{\lambda,1} = -\frac{1}{\alpha}\Psi(m) + \ln(\lambda) + (m-1)\frac{\ln(m)}{m\alpha} \quad (39)$$

and

$$\kappa_{\lambda,q} = (-1)^q \frac{1}{\alpha^q}\Psi^{(q-1)}(m) + (m-1)\left[\frac{\ln(m)}{m\alpha}\right]^q (q-1)! \quad (40)$$

where $\kappa_{\lambda,q}$ denotes the q-th cumulant of $\ln(\hat{\lambda})$, $q \geq 1$.

From (39) and (40) we obtain the following expressions for the bias and variance of $\ln(\hat{\lambda})$:

$$\ln(\lambda) - E\left[\ln(\hat{\lambda})\right] = \ln(\lambda) - \kappa_{\lambda,1} = \frac{1}{\alpha}\left[\Psi(m) - (m-1)\frac{\ln(m)}{m}\right] \quad (41)$$

and

$$Var\left[\ln(\hat{\lambda})\right] = \kappa_{\lambda,2} = \frac{1}{\alpha^2}\left[\Psi'(m) + (m-1)\left(\frac{\ln(m)}{m}\right)^2\right] \quad (42)$$

It is easy to show that (41) is equivalent to (12). This is a consequence of (14) and the fact that

$$\Psi(m) = \sum_{i=1}^{m-1}\frac{1}{i} - \gamma, \;\; m \geq 1 \quad (43)$$

where γ is the Euler-Mascheroni constant (see (13)). In turn, the derivative of the digamma function present in (42), known as the trigamma function, fulfils the following equality:

$$\Psi'(m) = \sum_{k=0}^{\infty}\frac{1}{(m+k)^2} \quad (44)$$

Equations (43) and (44), known from the special functions theory, allow to find close approximations of the expressions in square brackets in (41) and (42).

6. Approximating the biases and variances of $\widehat{A}$ and $\widehat{\Lambda}$

As follows from (30) and (41) – (44), the biases and variances of $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$ are expressed as functions of m and α. Since α is a parameter to be estimated, its exact value is unknown. However, α is equal to $(m-1)/mE(1/\hat{\alpha})$ (see (14)), where $E(1/\hat{\alpha})$ can be approximated by $\hat{A}$ defined in (7). In consequence, Equations (30), (41) and (42) adopt the following form:

$$1/\alpha - E(1/\hat{\alpha}) \approx \frac{1}{(m-1)}\hat{A} \quad (45)$$

$$Var(1/\hat{\alpha}) \approx \frac{1}{(m-1)}\left(\hat{A}\right)^2 \quad (46)$$

$$\ln(\lambda) - E\left[\ln(\hat{\lambda})\right] \approx \left[\frac{m\Psi(m)}{m-1} - \ln(m)\right]\hat{A} \quad (47)$$

$$Var\left[\ln(\hat{\lambda})\right] \approx \left[\frac{m^2}{m-1}\Psi'(m) + [\ln(m)]^2\right]\frac{(\hat{A})^2}{m-1} \quad (48)$$

As shown in the introduction (see the paragraph below (10)), the biases of $\hat{A}$ and $\widehat{\Lambda}$ are equal to those of $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$, hence their approximate values are the same as given in (45) and (47). In turn, according to (10), the approximate variances of $\hat{A}$ and $\widehat{\Lambda}$ are obtained by dividing the right-hand sides of (46) and (48) by n. The same symbol denotes an estimator defined as a random variable and one defined as a fixed value, but its actual meaning is clear from the context.

7. The accuracy of $\widehat{A}$ and $\widehat{\Lambda}$ in the context of confidence intervals

The central limit theorem says that N(0,1) is the limit distribution of the random variable $(\hat{X} - \mu)\sqrt{n}/\sigma$, where $\hat{X}$ is the sample mean of n independent realizations of a random variable X with finite expected value μ and variance σ^2. The limit is taken for $n \to \infty$. It thus follows that for sufficiently large n we have:

$$\Pr\left[-\varepsilon < (\hat{X} - \mu)\frac{\sqrt{n}}{\sigma} < \varepsilon\right] \approx 2\Phi(\varepsilon) - 1 \quad (49)$$

where Φ is the Gauss error function. Substituting ε with $\varepsilon\sqrt{n}/\sigma$ in (49) yields:

$$\Pr\left[-\varepsilon\frac{\sqrt{n}}{\sigma} < (\hat{X} - \mu)\frac{\sqrt{n}}{\sigma} < \varepsilon\frac{\sqrt{n}}{\sigma}\right] = \Pr\left[\hat{X} - \varepsilon < \mu < \hat{X} + \varepsilon\right] \approx 2\Phi\left(\varepsilon\frac{\sqrt{n}}{\sigma}\right) - 1 \quad (50)$$

Let $\hat{X}-\varepsilon$ and $\hat{X}+\varepsilon$ be regarded as the limits of the confidence interval for μ, at the confidence level c. This means that the probabilities in (50) are equal to c (usually, the confidence level is denoted as $1-\alpha$, where α is close to zero, but α denotes the shape parameter in this paper). In view of (50), the sample size for which the level c is reached is the smallest number n satisfying the following inequality:

$$2\Phi\left(\varepsilon\frac{\sqrt{n}}{\sigma}\right)-1\geq \mathrm{c} \tag{51}$$

From (51) we readily obtain the following condition to be satisfied by n so that $(\hat{X}-\varepsilon,\hat{X}+\varepsilon)$ is the confidence interval for μ, at the confidence level c:

$$n\geq\sigma^2\left[\Phi^{-1}\left(\frac{1+c}{2}\right)/\varepsilon\right]^2 \tag{52}$$

Given ε, c and σ, we can find n from the widely available tables of the Gauss error function. Clearly, it is essential to have σ in order to find the minimum sample size ensuring that the desired confidence level is attained. It only takes to replace σ^2 in (52) with the approximate variance of $1/\hat{\alpha}$ or $\ln(\hat{\lambda})$ to obtain the minimum n guaranteeing that $(\hat{A}-\varepsilon,\ \hat{A}+\varepsilon)$ or $(\hat{\Lambda}-\varepsilon,\ \hat{\Lambda}+\varepsilon)$ is the confidence interval for $E(1/\hat{\alpha})$ or $E[\ln(\hat{\lambda})]$ at the confidence level c. The required approximate variances are given by (46) and (48). However, as $E(1/\hat{\alpha})$ or $E[\ln(\hat{\lambda})]$ differ from $1/\alpha$ or $\ln(\lambda)$, the half-width of the confidence interval placed in (52) and/or the interval bounds have to be properly modified if n is computed for the confidence interval for $1/\alpha$ or $\ln(\lambda)$.

In view of (14) we have

$$\Pr\left[\hat{A}-\frac{m-1}{m}\varepsilon<E(1/\hat{\alpha})<\hat{A}+\frac{m-1}{m}\varepsilon\right]=\Pr\left[\hat{A}-\frac{m-1}{m}\varepsilon<\frac{m-1}{m\alpha}<\hat{A}+\frac{m-1}{m}\varepsilon\right]$$
$$=\Pr\left[\frac{m}{m-1}\hat{A}-\varepsilon<1/\alpha<\frac{m}{m-1}\hat{A}+\varepsilon\right] \tag{53}$$

where $\hat{A}$ is given by (7). As implied by (53) and the preceding argument, if the interval $(\hat{A}m/(m-1)-\varepsilon,\ \hat{A}m/(m-1)+\varepsilon)$ is to contain the parameter $1/\alpha$ at the assumed confidence level c, then n should satisfy the following condition:

$$n\geq Var(1/\hat{\alpha})\left[\Phi^{-1}\left(\frac{1+c}{2}\right)m/\varepsilon(m-1)\right]^2\approx\frac{m^2}{(m-1)^3}\left(\hat{A}\right)^2\left[\Phi^{-1}\left(\frac{1+c}{2}\right)/\varepsilon\right]^2 \tag{54}$$

The variance of $1/\hat{\alpha}$ is approximated using (46).

In turn, (12) yields:

$$\Pr[\hat{\Lambda}-\varepsilon<E[\ln(\hat{\lambda})]<\hat{\Lambda}+\varepsilon]=\Pr[\hat{\Lambda}-\varepsilon<\ln(\lambda)-\varphi(m)E(1/\hat{\alpha})<\hat{\Lambda}+\varepsilon]$$
$$=\Pr[\hat{\Lambda}+\varphi(m)E(1/\hat{\alpha})-\varepsilon<\ln(\lambda)<\hat{\Lambda}+\varphi(m)E(1/\hat{\alpha})+\varepsilon]$$
$$\approx\Pr[\hat{\Lambda}+\varphi(m)\cdot\hat{A}\ -\varepsilon<\ln(\lambda)<\hat{\Lambda}+\varphi(m)\cdot\hat{A}\ +\varepsilon] \tag{55}$$

where $\hat{\Lambda}$ is given by (8) and

$$\varphi(m)=\frac{m}{m-1}\left[\sum_{j=1}^{m-1}\frac{1}{j}-\frac{m-1}{m}\ln(m)-\gamma\right] \tag{56}$$

For the definition of γ see (13). It thus holds that $(\hat{\Lambda}+\varphi(m)\cdot\hat{A}\ -\varepsilon,\ \hat{\Lambda}+\varphi(m)\cdot\hat{A}\ +\varepsilon)$ is the confidence interval for $\ln(\lambda)$ at the confidence level c, if n satisfies the following condition:

$$n\geq Var[\ln(\hat{\lambda})]\left[\Phi^{-1}\left(\frac{1+c}{2}\right)/\varepsilon\right]^2 \tag{57}$$

The variance of $\mathbf{ln}(\boldsymbol{\lambda})$ should be approximated using (48).

8. Concluding remarks

This paper extends the results of (Malinowski, 2022) which presents a new approach to estimating the shape (α) and scale (λ) parameters of the Weibull distribution. According to this approach, each parameter is approximated by two estimators – the less accurate m-sample estimator based on a sequence of m times-to-failure of one test item undergoing m–1 minimal repairs, and the more accurate $n{\cdot}m$-sample estimator obtained by taking the mean value of n realizations of the m-sample one. The m-sample estimators of α and λ, i.e. $\hat{\alpha}$ and $\hat{\lambda}$, are given by (4). However, due to technical reasons explained in (Malinowski, 2022), we instead consider the m-sample estimators of $1/\alpha$ and $\ln(\lambda)$, i.e. $1/\hat{\alpha}$ and $\ln(\hat{\lambda})$ given by (5) and (6). Their biases can be computed from (11) and (12). The more accurate $n{\cdot}m$-sample estimators of $1/\alpha$ and $\ln(\lambda)$, i.e. $\hat{A}$ and $\hat{\Lambda}$ given by (7) and

(8), have the same biases as the ***m***-sample ones, but their variances are ***n*** times smaller.

Although the estimators $\hat{A}$ and $\widehat{\Lambda}$ approximate $1/\alpha$ and $\ln(\lambda)$, they can also be used to approximate α and λ, in view of the following equalities:

$$\alpha \approx \frac{m-1}{m\cdot\widehat{\mathrm{A}}} \tag{58}$$

$$\lambda \approx \exp\left[\widehat{\Lambda} + \varphi(m)\cdot\widehat{\mathrm{A}}\right] \tag{59}$$

where $\varphi(m)$ is defined by (56). The above two formulas follow from (11) and (12) if we take into account that

$$E(1/\hat{\alpha}) \approx \widehat{\mathrm{A}}, \quad E\left[\ln(\hat{\lambda}) \approx \widehat{\Lambda}\right] \tag{60}$$

for sufficiently large ***n***.

The accuracy of the estimators presented in (Malinowski, 2022) could not be established there because of the lack of formulas for the estimators' variances. Such formulas, namely (30) and (42), have been derived in the current paper. They include the unknown, yet to be estimated parameter $1/\alpha$, which can be approximated by $\widehat{\mathrm{A}}$, thus yielding (46) and (48). These formulas can be used to find the sample size ***n*** for which the confidence interval of a given width contains the parameter $1/\alpha$ or $\ln(\lambda)$ (and, in consequence, α or λ) at a given confidence level. How to do it is demonstrated in Section 7.

The future work should focus on analyzing the confidence intervals for the parameters α and λ (let us note that in Section 7 the intervals for $1/\alpha$ and $\ln(\lambda)$ are considered). Also, the analytical results should be illustrated by numerical examples based on data obtained from practice.

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